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Independent products of numerical possibility measures

Enrique Miranda
University of Oviedo
Department of Statistics and Operations Research
C/Calvo Sotelo s/n
33007 Oviedo, Spain
e-mail: alu426@pinon.ccu.uniovi.es

Gert de Cooman
Universiteit Gent
Onderzoeksgroep SYSTeMS
Technologiepark – Zwijnaarde 9
9052 Zwijnaarde, Belgium
e-mail: gert.decooman@rug.ac.be

Abstract

Possibility measures can be given a behavioural interpretation as systems of upper betting rates. As such, they should arguably satisfy certain rationality requirements. Using a version of Walley's notion of epistemic independence suitable for possibility measures, we investigate what these requirements tell us about the construction of independent product possibility measures from given marginals.

Keywords. Possibility theory, upper probability, coherence, conditioning, epistemic independence, independent product.

1 Introduction

Possibility theory, as originated by Zadeh [17], can be described as collection of notions and techniques centered around the notion of a *possibility measure*. It is mainly used for the representation and manipulation of so-called *linguistic uncertainty*, produced by (potentially vague) statements in natural language. It was conceived as an alternative to probability theory, which, according to Zadeh, does not lend itself very well to modelling linguistic uncertainty. In parallel with probability theory, notions such as possibility integrals, product possibility measures, conditional possibility measures and possibilistic independence have been developed [2, 6, 7]. Possibility measures have also been studied under different names and guises, and in other contexts, see for instance [1, 9, 10, 11, 12].

In recent years, quite some effort has been invested in the study of possibility measures in the framework of the theory of imprecise probabilities [13]. In this approach, the possibility of some event is given the behavioural interpretation of a subject's *upper probability*, or upper betting rate, for the event, i.e., the infimum rate at which the subject is willing to take bets on the event. A possibility measure then represents a collection of such upper betting rates. Because specifying an upper betting rate amounts to a commitment to act (bet) in certain ways, upper probabilities and in particular possibility measures are subject to a number of rationality, or consistency, requirements, called *avoiding sure loss* and *coherence*. It turns out that *normal* possibility measures satisfy these requirements, and can therefore be considered as reasonable imprecise probability models [3, 4, 5, 14]. So can (precise) probability measures. This points to a distinct advantage of the unifying approach using the theory of imprecise probabilities: it allows the comparison of both types of measures in a single framework, using a common language and the same (behavioural) interpretation. This has for instance been done in a recent study [16], where it is argued that possibility measures indeed seem to be better suited for modelling linguistic uncertainty than probability measures.

This being said, it is by no means obvious that all of what is commonly understood as 'possibility theory' will get similar backing from the theory of imprecise probabilities: the rationality criteria of avoiding sure loss and coherence can for instance be used to weed out those notions and techniques which are inconsistent with the behavioural interpretation of possibility measures as upper probabilities. To give an example, in contradistinction to probability theory, a large variety of rules have been proposed for conditioning a possibility measure (see for instance the overviews in [2, 7, 15]). In a recent paper [15], Walley and De Cooman have shown that most of these rules avoid sure loss, but do not satisfy the stricter requirement of coherence. They have also suggested a number of new conditioning rules that guarantee coherence.

Two variables are said to be *epistemically independent* to a subject when new knowledge about the value that one variable assumes, does not change his beliefs about the value the other variable takes [13, Chapter 9]. In the present paper, we study some aspects of this notion of independence for possibility measures. More specifically, we investigate what the rationality criteria of avoiding sure loss and coherence tell us about the construction of independent joint possibility measures from given marginal ones.

We have organised the paper as follows. In Section 2, we briefly review definitions and basic results concerning the interpretation of possibility measures as upper betting rates, necessary for understanding much of what follows. In Section 3, we formulate a definition of epistemic independence inspired by Walley's original definition [13], and suitable in a 'possibilistic' context. We also derive a necessary and sufficient condition, in terms of sets of dominated probability measures, for the consistency of a joint possibility measure with its marginals, under the epistemic independence assumption. This condition is quite complicated, but we show in Section 4 that it can be simplified significantly when one of the marginal possibility measures is unimodal: we obtain a characterisation of the coherent product possibility measures through an upper bound. The study for the plurimodal case seems to be much harder, and we present a simplified sufficient, and a different necessary, condition for coherence under the epistemic independence assumption in Section 5. Section 6 concludes the paper with additional discussion.

2 Preliminary notions and results

A *possibility measure* Π on a finite[1] set Ω is a map defined on the power set $\wp(\Omega)$ of Ω and taking values in the real unit interval $[0,1]$, that satisfies $\Pi(\emptyset)=0$ and that is moreover *maxitive*: for all subsets A and B of Ω, $\Pi(A\cup B)=\max\{\Pi(A),\Pi(B)\}$. It is completely determined by its *(possibility) distribution* $\pi\colon\Omega\to[0,1]$, defined by $\pi(\omega)=\Pi(\{\omega\})$ for all $\omega\in\Omega$. Indeed, we have $\Pi(A)=\max\{\pi(\omega)\colon\omega\in A\}$ for any non-empty $A\subseteq\Omega$.

Possibility measures can be incorporated into the behavioural theory of imprecise probabilities [13] by interpreting them as upper probabilities: for any event $A\subseteq\Omega$, $\Pi(A)$ is then a subject's upper probability of A, i.e., his infimum acceptable rate for *taking* bets *on* A, or one minus his supremum acceptable rate for betting *against* A. This means that the subject is disposed to accept a bet whose outcome is $x-1$ if A occurs, and x if A doesn't occur, for all $x>\Pi(A)$. It turns out [3, 4, 14] that a possibility measure Π with this interpretation satisfies the rationality criteria of avoiding sure loss and coherence[2] if and only if it is *normal*, i.e., if $\Pi(\Omega)=1$. We shall therefore only consider normal possibility measures in what follows. Normality implies that the distribution π has at least one *mode* (or modal value) ω_o, for which $\pi(\omega_o)=1$. If there is only one such mode, then π (and Π) is called *unimodal*. A distribution with more than one mode is called *plurimodal*.

Consider two variables X and Y taking values in the respective finite sets $\mathcal{X}$ and $\mathcal{Y}$. We only consider the interesting case that both $\mathcal{X}$ and $\mathcal{Y}$ have more than one element. We assume that a subject has certain beliefs about which values these variables assume, and that he models these beliefs using a possibility measure $\Pi_{X,Y}$ on $\mathcal{X}\times\mathcal{Y}$, with distribution $\pi_{X,Y}$.[3] For $C\subseteq\mathcal{X}\times\mathcal{Y}$, $\Pi_{X,Y}(C)$ is the subject's upper probability for the event that (X,Y) assumes a value in C, and for $(x,y)\in\mathcal{X}\times\mathcal{Y}$, $\pi_{X,Y}(x,y)$ is his upper probability that (X,Y) assumes the value (x,y).

The *marginals* Π_X and Π_Y of the so-called *joint* possibility measure $\Pi_{X,Y}$ are defined as follows. Π_X is a possibility measure on $\mathcal{X}$, and $\Pi_X(A)=\Pi_{X,Y}(A\times\mathcal{Y})$ is the subject's upper probability that the variable X assumes a value in $A\subseteq\mathcal{X}$ (regardless of what value Y takes); similarly, Π_Y is a possibility measure on $\mathcal{Y}$ and $\Pi_Y(B)=\Pi_{X,Y}(\mathcal{X}\times B)$ is his upper probability that Y takes a value in $B\subseteq\mathcal{Y}$. We denote the possibility distributions of Π_X and Π_Y by π_X and π_Y respectively.

Conditional possibility measures [2, 3, 6, 7] can be given the behavioural interpretation of *updated* upper probabilities [3, 15]. $\Pi_{X|Y}(A|y)$ is then interpreted as a subject's infimum acceptable rate for taking bets on the event that X assumes a value in $A\subseteq\mathcal{X}$, after learning only that Y takes the value $y\in\mathcal{Y}$; and similarly for $\Pi_{Y|X}(B|x)$. For each $x\in\mathcal{X}$, $\Pi_{Y|X}(\cdot|x)$ is assumed to be a possibility measure on $\mathcal{Y}$, with distribution $\pi_{Y|X}(\cdot|x)$; and for each $y\in\mathcal{Y}$, $\Pi_{X|Y}(\cdot|y)$ is assumed to be a possibility measure on $\mathcal{X}$, with distribution $\pi_{X|Y}(\cdot|y)$.[4]

Since on a behavioural interpretation, the joint and the conditional possibility measures represent a subject's dispositions to act in certain ways, they should satisfy certain rationality requirements, not only separately (they should all be normal!) but *also taken together*. A thorough discussion of such criteria in the general context of imprecise probabilities was given by Walley [13]. The special case of possibility measures was discussed by Walley and De Cooman [15], who also investigated which of a large number of so-called conditioning rules for possibility measures, available in the literature, satisfy these criteria. We refer to their work for both motivation and mathematical development. For the purposes of the present paper, it will suffice to recall the following characterisation of the criteria of avoiding sure loss and of coherence[5] of the joint and conditional possibility distributions (or equivalently, measures) in terms of sets of dominated probability measures. It can be easily inferred from Lemma 3 and the proof of Theorem 1 in [15]. Let $\mathcal{M}_c$ be the set of probability measures defined on the power set of $\mathcal{X}\times\mathcal{Y}$ and satisfying the following inequalities:

(C_1) $P(A)\leq\Pi_{X,Y}(A)$ for all $A\subseteq\mathcal{X}\times\mathcal{Y}$; and

[1] We only deal with possibility measures on *finite* sets in this paper.

[2] We assume that the reader is familiar with these basic consistency requirements in the theory of imprecise probabilities. See [3, 4, 13, 14] for more details.

[3] A specific and interesting case where this assumption makes sense, is discussed in [16].

[4] We only consider the case that the subject's conditional upper probabilities are possibility measures as well. This is perfectly compatible with the epistemic independence assumption to be introduced and studied later.

[5] Walley [13, Section 7.1] speaks of *avoiding uniform sure loss* and of *weak coherence*; see also Technical Remark 2 in [15].

(C_2) $P(B \times \{y\})/P(\mathcal{X} \times \{y\}) \leq \Pi_{X|Y}(B|y)$ for all $B \subseteq \mathcal{X}$ and $y \in \mathcal{Y}$ such that $P(\mathcal{X} \times \{y\}) > 0$; and

(C_3) $P(\{x\} \times C)/P(\{x\} \times \mathcal{Y}) \leq \Pi_{Y|X}(C|x)$ for all $C \subseteq \mathcal{Y}$ and $x \in \mathcal{X}$ such that $P(\{x\} \times \mathcal{Y}) > 0$.

Theorem 1. *The joint possibility distribution $\pi_{X,Y}$ and the conditional possibility distributions $\{\pi_{Y|X}(\cdot|x) \colon x \in \mathcal{X}\}$ and $\{\pi_{X|Y}(\cdot|y) \colon y \in \mathcal{Y}\}$ avoid sure loss if and only if $\mathcal{M}_c$ is non-empty. They are coherent if and only if there is a non-empty set $\mathcal{M}$ of probabilities defined on the power set of $\mathcal{X} \times \mathcal{Y}$ such that:*

1. $\Pi_{X,Y}(A) = \sup\{P(A) \colon P \in \mathcal{M}\}$ *for all* $A \subseteq \mathcal{X} \times \mathcal{Y}$.
2. $\Pi_{X|Y}(B|y) \geq \sup\{P(B \times \{y\})/P(\mathcal{X} \times \{y\}) \colon P \in \mathcal{M}, P(\mathcal{X} \times \{y\}) > 0\}$ *for all* $B \subseteq \mathcal{X}$ *and* $y \in \mathcal{Y}$, *with equality when* $\beta(y) = \max\{\pi_Y(v) \colon v \neq y\} < 1$.
3. $\Pi_{Y|X}(C|x) \geq \sup\{P(\{x\} \times C)/P(\{x\} \times \mathcal{Y}) \colon P \in \mathcal{M}, P(\{x\} \times \mathcal{Y}) > 0\}$ *for all* $C \subseteq \mathcal{Y}$ *and* $x \in \mathcal{X}$, *with equality when* $\eta(x) = \max\{\pi_X(u) \colon u \neq x\} < 1$.

If there is such a set $\mathcal{M}$, then $\mathcal{M}_c$ is the largest such set.

A simple *necessary* condition for the coherence of $\pi_{X,Y}$, $\{\pi_{Y|X}(\cdot|x) \colon x \in \mathcal{X}\}$ and $\{\pi_{X|Y}(\cdot|y) \colon y \in \mathcal{Y}\}$ was shown in [15] to be the following:

$$\pi_{X,Y}(x,y) \leq \frac{\pi_{X|Y}(x|y)\pi_{Y|X}(y|x)\max\{\pi_X(x), \pi_Y(y)\}}{\pi_{X|Y}(x|y) + \pi_{Y|X}(y|x) - \pi_{X|Y}(x|y)\pi_{Y|X}(y|x)} \tag{1}$$

for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, where $\frac{0}{0}$ is taken to be 0.

3 Epistemic independence and coherence

We are now ready to address the question that will occupy us in the rest of the paper. Assume that our subject has beliefs (or information) about the values assumed by the variables X and Y separately, and that he has modelled his beliefs in the form of the marginal possibility distributions π_X and π_Y. He also judges the variables X and Y to be epistemically[6] independent: he judges that new information about the value of one variable will not affect his beliefs about the value the other variable assumes. We intend to investigate what this independence assumption, together with the rationality requirements of avoiding sure loss and coherence, tells us about the joint distribution $\pi_{X,Y}$, which models the subject's beliefs about the values X and Y assume jointly.

[6] There is more than one independence concept in possibility theory, see for instance [2, 7]. Here, we use a version of Walley's notion of epistemic independence [13], because it has the most natural interpretation in the behavioural context of the theory of imprecise probabilities.

For probability measures (on finite spaces), the judgement of epistemic independence together with coherence leads uniquely to the product probability measure of the marginals [13, Section 9.3.2]. We shall see that there is no uniqueness in the case of possibility measures: for given marginals, there is generally more than one joint possibility distribution that satisfies the independence and coherence requirements. Our aim is to characterise such joint distributions in a manner that is as simple as possible.

The first step we have to take is to apply the notion of epistemic independence, formulated by Walley for general imprecise models [13, Section 9] to the case that beliefs are represented by possibility distributions.

Definition 1. We say that Y is *irrelevant* to X when $\pi_{X|Y}(x|y) = \pi_X(x)$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. We say that X and Y are *epistemically independent* when X is irrelevant to Y and Y is irrelevant to X.

Given the marginal distributions π_X and π_Y, the judgement of epistemic independence leads at once to values for the conditional distributions $\{\pi_{X|Y}(\cdot|y) \colon y \in \mathcal{Y}\}$ and $\{\pi_{Y|X}(\cdot|x) \colon x \in \mathcal{X}\}$. We now only have to require that the joint $\pi_{X,Y}$ (which has marginals π_X and π_Y) should be consistent with these conditional distributions.

Definition 2. We say that the normal joint possibility distribution $\pi_{X,Y}$ *avoids sure loss under epistemic independence* when the joint distribution and the conditional possibility distributions $\{\pi_{Y|X}(\cdot|x) \colon x \in \mathcal{X}\}$ and $\{\pi_{X|Y}(\cdot|y) \colon y \in \mathcal{Y}\}$ given by

$$\pi_{X|Y}(x|y) = \pi_X(x) \quad \text{and} \quad \pi_{Y|X}(y|x) = \pi_Y(y) \tag{2}$$

for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, avoid sure loss. Similarly, we say that the joint distribution $\pi_{X,Y}$ is *coherent under epistemic independence* when these possibility distributions are coherent. In that case, $\pi_{X,Y}$ will be called an *independent joint distribution*, or an *independent product* of its marginals π_X and π_Y.

It turns out that the first consistency condition under epistemic independence is always satisfied. The second condition is more involved, however. To see this, consider the set $\mathcal{M}_i$ (the counterpart of the set $\mathcal{M}_c$ in the previous section) of probability measures defined on the power set of $\mathcal{X} \times \mathcal{Y}$ and satisfying the following inequalities:

(CI_1) $P(A) \leq \Pi_{X,Y}(A)$ for all $A \subseteq \mathcal{X} \times \mathcal{Y}$; and

(CI_2) $P(B \times \{y\})/P(\mathcal{X} \times \{y\}) \leq \Pi_X(B)$ for all $B \subseteq \mathcal{X}$ and $y \in \mathcal{Y}$ such that $P(\mathcal{X} \times \{y\}) > 0$; and

(CI_3) $P(\{x\} \times C)/P(\{x\} \times \mathcal{Y}) \leq \Pi_Y(C)$ for all $C \subseteq \mathcal{Y}$ and $x \in \mathcal{X}$ such that $P(\{x\} \times \mathcal{Y}) > 0$.

Applying Theorem 1 leads to the following result, which is the starting point for the further development.

Theorem 2. *A normal joint possibility distribution $\pi_{X,Y}$ always avoids sure loss under epistemic independence, or in other words, $\mathcal{M}_i \neq \emptyset$. It is coherent under epistemic independence if and only if there is a non-empty set of probabilities $\mathcal{M}$ defined on the power set of $\mathcal{X} \times \mathcal{Y}$ such that:*

1. $\Pi_{X,Y}(A) = \sup\{P(A) \colon P \in \mathcal{M}\}$ *for all* $A \subseteq \mathcal{X} \times \mathcal{Y}$.
2. $\Pi_X(B) \geq \sup\{P(B \times \{y\})/P(\mathcal{X} \times \{y\}) \colon P \in \mathcal{M}, P(\mathcal{X} \times \{y\}) > 0\}$ *for all* $B \subseteq \mathcal{X}$ *and* $y \in \mathcal{Y}$, *with equality when* $\beta(y) < 1$.
3. $\Pi_Y(C) \geq \sup\{P(\{x\} \times C)/P(\{x\} \times \mathcal{Y}) \colon P \in \mathcal{M}, P(\{x\} \times \mathcal{Y}) > 0\}$ *for all* $C \subseteq \mathcal{Y}$ *and* $x \in \mathcal{X}$, *with equality when* $\eta(x) < 1$.

If there is such a set $\mathcal{M}$, then $\mathcal{M}_i$ is the greatest such set.

Proof. The coherence part follows immediately from Theorem 1. The same theorem tells us that $\pi_{X,Y}$ avoids sure loss under epistemic independence if and only if $\mathcal{M}_i \neq \emptyset$. It therefore only remains to be shown that $\mathcal{M}_i \neq \emptyset$. Consider $(x, y) \in \mathcal{X} \times \mathcal{Y}$ such that $\pi_{X,Y}(x, y) = 1$, and consequently $\pi_X(x) = \pi_Y(y) = 1$ (there always are such x and y, since $\pi_{X,Y}$ is normal). Define the (degenerate) probability measure P on the power set of $\mathcal{X} \times \mathcal{Y}$ by $P(x, y) = 1$. Then it is easy to see that $P \in \mathcal{M}_i$. □

We can also take a look at the necessary condition for coherence (1), mentioned in the previous section. Using the epistemic independence relation (2), we find:

$$\pi_{X,Y}(x, y) \leq \frac{\pi_X(x)\pi_Y(y) \max\{\pi_X(x), \pi_Y(y)\}}{\pi_X(x) + \pi_Y(y) - \pi_X(x)\pi_Y(y)} \quad \text{(NC)}$$

for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, where $\frac{0}{0}$ is taken to be 0. This is a very simple *necessary* condition for the coherence under epistemic independence of $\pi_{X,Y}$, expressed only in terms of the *local* values $\pi_{X,Y}(x, y)$, $\pi_X(x)$ and $\pi_Y(y)$ of the joint distribution and its marginals. We can easily deduce from this condition certain properties that will be used repeatedly further on. Their proof is fairly straightforward, and is therefore omitted.

Lemma 3. *If the normal joint distribution $\pi_{X,Y}$ satisfies the necessary condition* (NC)*, then for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$:*

1. $\pi_{X,Y}(x, y) \leq \pi_X(x)\pi_Y(y)$;
2. *if* $0 < \pi_{X,Y}(x, y) = \pi_X(x)$ *then* $\pi_Y(y) = 1$;
3. *if* π_Y *is unimodal with unique mode* y_o, *then* $\pi_X(x) = \pi_{X,Y}(x, y_o)$.
4. *if* $0 < \pi_X(x) < 1$ *and* $0 < \pi_Y(y) < 1$ *then* $\pi_{X,Y}(x, y) < \pi_X(x)\pi_Y(y)$.

In the rest of this section, we investigate how the necessary and sufficient condition of Theorem 2 can be simplified. Our efforts will culminate in Theorem 8, which is the most important stepping stone for our investigation in the following sections. First of all, in checking the coherence condition, the following lemma will be very useful, because it helps us verify whether a probability measure belongs to $\mathcal{M}_i$ or not. The proof is elementary, and therefore omitted.

Lemma 4. *Let m be the number of elements in $\mathcal{X}$, and n the number of elements in $\mathcal{Y}$. Consider a probability measure P defined on the power set of $\mathcal{X} \times \mathcal{Y}$.*

1. *Assume that the mn elements $z = (x, y)$ of $\mathcal{X} \times \mathcal{Y}$ are labeled in such a way that $\pi_{X,Y}(z_1) \leq \pi_{X,Y}(z_2) \leq \cdots \leq \pi_{X,Y}(z_{mn})$. Then P satisfies condition (CI_1) if and only if $P(z_1) + \cdots + P(z_j) \leq \pi_{X,Y}(z_j)$ for $j = 1, \ldots, nm$.*
2. *Assume that the m elements of $\mathcal{X}$ are labeled in such a way that $\pi_X(x_1) \leq \pi_X(x_2) \leq \cdots \leq \pi_X(x_m)$. Then P satisfies condition (CI_2) if and only if for all $y \in \mathcal{Y}$ such that $P(\mathcal{X} \times \{y\}) > 0$ and for $j = 1, \ldots, m$,*

$$\frac{P(x_1, y) + \cdots + P(x_j, y)}{P(\mathcal{X} \times \{y\})} \leq \pi_X(x_j).$$

3. *Assume that the n elements of $\mathcal{Y}$ are labeled in such a way that $\pi_Y(y_1) \leq \pi_Y(y_2) \leq \cdots \leq \pi_Y(y_n)$. Then P satisfies condition (CI_3) if and only if for all $x \in \mathcal{X}$ such that $P(\{x\} \times \mathcal{Y}) > 0$ and for $j = 1, \ldots, n$,*

$$\frac{P(x, y_1) + \cdots + P(x, y_j)}{P(\{x\} \times \mathcal{Y})} \leq \pi_Y(y_j).$$

Interestingly, coherence under independence is not influenced by removing from the set $\mathcal{X}$ elements x such that $\pi_X(x) = 0$ and from the set $\mathcal{Y}$ elements y such that $\pi_Y(y) = 0$.[7] To see this, consider the marginal sets

$$\mathcal{X}' = \{x \in \mathcal{X} \colon \pi_X(x) > 0\}$$
$$\mathcal{Y}' = \{y \in \mathcal{Y} \colon \pi_Y(y) > 0\}$$

and denote by $\Pi'_{X,Y}$ the restriction of $\Pi_{X,Y}$ to the power set of $\mathcal{X}' \times \mathcal{Y}'$. With this (normal) possibility measure, with possibility distribution $\pi'_{X,Y}$, we may associate a set $\mathcal{M}'_i$ of probability measures on the power set of $\mathcal{X}' \times \mathcal{Y}'$ satisfying the (corresponding) properties (CI_1)–(CI_3), which by Theorem 2 completely determines the coherence under independence of the joint distribution $\pi'_{X,Y}$ (or the possibility measure $\Pi'_{X,Y}$).

[7] For our subject, it is practically impossible that the variables X and Y assume such values, since he is disposed to bet *at all odds* against the event that they do.

Proposition 5. *$\mathcal{M}_i$ satisfies the conditions of Theorem 2 if and only if $\mathcal{M}'_i$ satisfies them, or in other words, the normal joint distribution $\pi_{X,Y}$ is coherent under independence if and only if $\pi'_{X,Y}$ is.*

Proof. The proof is immediate if we observe that the elements of $\mathcal{M}_i$ and those of $\mathcal{M}'_i$ are in one-to-one correspondence, and that $\mathcal{M}'_i$ consists of the restrictions to $\mathcal{X}' \times \mathcal{Y}'$ of the probabilities in $\mathcal{M}_i$. $\square$

This implies that our results will remain valid if, instead of using condition (2) to define epistemic independence, we use the alternative condition:

$$\pi_{X|Y}(x|y) = \pi_X(x) \text{ if } \pi_Y(y) > 0$$
$$\pi_{Y|X}(y|x) = \pi_Y(y) \text{ if } \pi_X(x) > 0$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, which is sometimes found in the literature (see for instance [8]).

Proposition 6. *The set of probabilities $\mathcal{M}_i$ satisfies the first condition of Theorem 2 if and only if for all (x, y) in $\mathcal{X} \times \mathcal{Y}$ there is a P in $\mathcal{M}_i$ such that $P(x, y) = \pi_{X,Y}(x, y)$.*

Proof. We first show that the condition is sufficient. Indeed, for any $A \subseteq \mathcal{X} \times \mathcal{Y}$, there is some $(x_A, y_A) \in A$ such that $\Pi_{X,Y}(A) = \pi_{X,Y}(x_A, y_A)$, and the condition tells us moreover that there is some $P \in \mathcal{M}_i$ such that $P(x_A, y_A) = \pi_{X,Y}(x_A, y_A)$, whence $\Pi_{X,Y}(A) \leq P(A)$. Since for all $Q \in \mathcal{M}_i$, condition (CI_1) tells us that $Q(A) \leq \Pi_{X,Y}(A)$, we infer that $\Pi_{X,Y}(A) = \max\{Q(A) \colon Q \in \mathcal{M}_i\}$. Next, we show that the condition is necessary. Consider $(x, y) \in \mathcal{X} \times \mathcal{Y}$. If $\mathcal{M}_i$ satisfies the first condition of Theorem 2, then $\pi_{X,Y}(x, y) = \sup\{P(x, y) \colon P \in \mathcal{M}_i\}$. Since $\mathcal{M}_i$ is obviously closed in the natural topology, this supremum is actually achieved for some $P \in \mathcal{M}_i$, or in other words, there is some $P \in \mathcal{M}_i$ such that $P(x, y) = \pi_{X,Y}(x, y)$. $\square$

Proposition 7. *If the normal joint distribution $\pi_{X,Y}$ satisfies the necessary condition* (NC)*, then the set $\mathcal{M}_i$ always satisfies the second and third conditions of Theorem 2.*

Proof. We show that $\mathcal{M}_i$ satisfies the second condition. The proof for the third condition is completely similar (or symmetrical). It follows from (CI_2) that we need only prove that for all $B \subseteq \mathcal{X}$:

$$\Pi_X(B) = \sup\{\frac{P(B \times \{y\})}{P(\mathcal{X} \times \{y\})} \colon P \in \mathcal{M}_i, P(\mathcal{X} \times \{y\}) > 0\},$$

when $\beta(y) < 1$. Let us suppose, therefore, that $\beta(y_o) < 1$, or in other words that π_Y is unimodal with unique mode y_o. Consider $B \subseteq \mathcal{X}$. Then there is some $x_B \in B$ such that $\Pi_X(B) = \pi_X(x_B)$. If $\Pi_X(B) = 1$, it follows from Lemma 3 and the unimodality of π_Y that $1 = \pi_X(x_B) = \pi_{X,Y}(x_B, y_o)$. The probability P uniquely defined on the power set of $\mathcal{X} \times \mathcal{Y}$ by $P(x_B, y_o) = 1$ is easily shown to belong to $\mathcal{M}_i$ and to attain the desired equality. Let us therefore consider the case that $\Pi_X(B) < 1$. Let x' be a modal point of the marginal distribution π_X. Note that $x' \notin B$ so $x' \neq x_B$. As π_Y is unimodal with unique mode y_o, we must have that $\pi_{X,Y}(x', y_o) = 1$. We also infer from Lemma 3 that $\pi_{X,Y}(x_B, y_o) = \pi_X(x_B)$. Consider the probability measure P uniquely defined on the power set of $\mathcal{X} \times \mathcal{Y}$ by $P(x_B, y_o) = \pi_{X,Y}(x_B, y_o)$ and $P(x', y_o) = 1 - \pi_{X,Y}(x_B, y_o)$. We proceed to show that $P \in \mathcal{M}_i$. Observe that $\pi_{X,Y}(x_B, y_o) < \pi_{X,Y}(x', y_o) = 1$, so Lemma 4 tells us that P satisfies (CI_1) if and only if $P(x_B, y_o) \leq \pi_{X,Y}(x_B, y_o)$, which holds by construction. Next, observe that $\pi_X(x_B) < \pi_X(x') = 1$. Since

$$\frac{P(x_B, y_o)}{P(\mathcal{X} \times \{y_o\})} = \frac{\pi_{X,Y}(x_B, y_o)}{1} \leq \pi_X(x_B)$$

and $P(\mathcal{X} \times \{y\}) = 0$ for every $y \in \mathcal{Y} \setminus \{y_o\}$, we may infer from Lemma 4 that P satisfies (CI_2). Since moreover $\pi_Y(y_o) = 1$ we immediately infer from Lemma 4 that P satisfies (CI_3) as well. We may therefore indeed conclude that $P \in \mathcal{M}_i$. It is now obvious that

$$\frac{P(B \times \{y_o\})}{P(\mathcal{X} \times \{y_o\})} = \frac{P(x_B, y_o)}{P(\mathcal{X} \times \{y_o\})} = \frac{\pi_X(x_B)}{1} = \Pi_X(B),$$

so the second condition of Theorem 2 is satisfied. $\square$

We may summarise these results in the following theorem.

Theorem 8. *The normal joint distribution $\pi_{X,Y}$ is coherent under independence if and only if it satisfies* (NC) *and if for all (x, y) in $\mathcal{X} \times \mathcal{Y}$ there is some P in $\mathcal{M}_i$ such that $P(x, y) = \pi_{X,Y}(x, y)$.*

In checking whether the conditions of this theorem are verified, the following lemma will allow us to proceed somewhat faster.

Lemma 9. *Assume that the normal joint distribution $\pi_{X,Y}$ satisfies condition* (NC) *and let (x, y) be an element of $\mathcal{X} \times \mathcal{Y}$ such that one of the following conditions is satisfied:*

1. $\pi_{X,Y}(x, y) = 0$*;*
2. $\max\{\pi_X(x), \pi_Y(y)\} = 1$*;*
3. $0 < \pi_{X,Y}(x, y)$ *and* $\max\{\pi_X(x), \pi_Y(y)\} < 1$*, and there are $x' \in \mathcal{X}$ and $y' \in \mathcal{Y}$ such that $\pi_{X,Y}(x', y) = \pi_Y(y)$, $\pi_{X,Y}(x, y') = \pi_X(x)$ and $\pi_{X,Y}(x', y') = 1$.*

Then there is a P in $\mathcal{M}_i$ such that $P(x, y) = \pi_{X,Y}(x, y)$.

Proof. Assume that the first condition is satisfied. We know from the first part of Theorem 2 that $\mathcal{M}_i \neq \emptyset$. It follows from condition (CI_1) and $\pi_{X,Y}(x, y) = 0$ that $P(x, y) = \pi_{X,Y}(x, y) = 0$ for all $P \in \mathcal{M}_i$.

Next, if the second condition holds, we may assume without loss of generality that $\pi_Y(y) = 1$. If $\pi_{X,Y}(x, y) = 1$, consider the (degenerate) probability measure defined on the power set of $\mathcal{X} \times \mathcal{Y}$ by $P(x, y) = 1 = \pi_{X,Y}(x, y)$. It is easily verified that $P \in \mathcal{M}_i$. If $\pi_{X,Y}(x, y) < 1$, then there is some $x' \neq x$ in $\mathcal{X}$ such that $\pi_{X,Y}(x', y) = 1$. Consider the probability measure P uniquely defined on the power set of $\mathcal{X} \times \mathcal{Y}$ by $P(x, y) = \pi_{X,Y}(x, y)$ and $P(x', y) = 1 - \pi_{X,Y}(x, y)$. It remains to be shown that $P \in \mathcal{M}_i$. First of all, recall that $\pi_{X,Y}(x, y) \leq \pi_{X,Y}(x', y) = 1$, so to prove that P satisfies (CI_1), Lemma 4 tells us that we need only verify that $P(x, y) \leq \pi_{X,Y}(x, y)$, which holds by construction. Next, observe that $\pi_X(x) \leq \pi_X(x') = 1$ and that $P(\mathcal{X} \times \{v\}) > 0$ only if $v = y$, so in order to verify that P satisfies (CI_2), Lemma 4 tells us that we need only verify that $P(x, y)/P(\mathcal{X} \times \{y\}) \leq \pi_X(x)$, or equivalently, $\pi_{X,Y}(x, y)/1 \leq \pi_X(x)$, which holds trivially. Finally, since $P(u, v) > 0$ only if $v = y$, and since $\pi_Y(y) = 1$, we infer from Lemma 4 that P also satisfies (CI_3), so indeed $P \in \mathcal{M}_i$.

To conclude the proof, let us assume that the third condition holds. Lemma 3 then tells us that $\pi_{X,Y}(x, y) < \pi_X(x)\pi_Y(y)$. Consequently, there is some $\alpha \in (0, 1)$ such that $\pi_{X,Y}(x, y) = \alpha\pi_X(x)\pi_Y(y)$. The same lemma also allows us to deduce that $\pi_X(x') = \pi_Y(y') = 1$ and therefore $x' \neq x$ and $y' \neq y$. We now define the finitely additive set function P on the power set of $\mathcal{X} \times \mathcal{Y}$ by:

$$\begin{aligned}
P(x, y) &= \pi_{X,Y}(x, y) \\
P(x, y') &= \alpha\pi_X(x) - \pi_{X,Y}(x, y) \\
P(x', y) &= \alpha\pi_Y(y) - \pi_{X,Y}(x, y) \\
P(x', y') &= 1 - \alpha[\pi_X(x) + \pi_Y(y)] + \pi_{X,Y}(x, y)
\end{aligned}$$

and $P(u, v) = 0$ for all other $(u, v) \in \mathcal{X} \times \mathcal{Y}$. We show that P is a probability. It is clear that $P(x, y) + P(x, y') + P(x', y) + P(x', y') = 1$, so it remains to be shown that all these terms are non-negative. First of all, it is obvious that $P(x, y) = \pi_{X,Y}(x, y) \geq 0$. Moreover

$$P(x', y) = \pi_{X,Y}(x, y)(\frac{1}{\pi_X(x)} - 1) \geq 0$$

and from the symmetry, we infer that also $P(x, y') \geq 0$. Finally, since

$$\begin{aligned}
&\alpha[\pi_X(x) + \pi_Y(y)] - \pi_{X,Y}(x, y) \\
&\quad = \pi_{X,Y}(x, y)(\frac{1}{\pi_X(x)} + \frac{1}{\pi_Y(y)} - 1) \\
&\quad = \pi_{X,Y}(x, y)\frac{\pi_X(x) + \pi_Y(y) - \pi_X(x)\pi_Y(y)}{\pi_X(x)\pi_Y(y)} \\
&\quad \leq 1,
\end{aligned}$$

where the last inequality follows from (NC), we see that $P(x', y') = 1 - \alpha[\pi_X(x) + \pi_Y(y)] + \pi_{X,Y}(x, y) \geq 0$. The proof is complete if we can show that $P \in \mathcal{M}_i$. We use Lemma 4. We may assume without loss of generality that $\pi_X(x) \leq \pi_Y(y)$, whence $\pi_{X,Y}(x, y) < \pi_{X,Y}(x, y') \leq \pi_{X,Y}(x', y) < \pi_{X,Y}(x', y') = 1$. Clearly, $P(x, y) = \pi_{X,Y}(x, y)$ and $P(x, y) + P(x, y') = \alpha\pi_X(x) \leq \pi_X(x) = \pi_{X,Y}(x, y')$. Moreover,

$$\begin{aligned}
&P(x, y) + P(x, y') + P(x', y) \\
&\quad = \alpha[\pi_X(x) + \pi_Y(y)] - \pi_{X,Y}(x, y) \\
&\quad = \pi_{X,Y}(x, y)\frac{\pi_X(x) + \pi_Y(y) - \pi_X(x)\pi_Y(y)}{\pi_X(x)\pi_Y(y)} \\
&\quad \leq \pi_Y(y) = \pi_{X,Y}(x', y),
\end{aligned}$$

where the last inequality follows from (NC). We may therefore conclude from Lemma 4 that P satisfies (CI_1). Next, observe that $\pi_X(x) < \pi_X(x') = 1$,

$$\frac{P(x, y)}{P(\mathcal{X} \times \{y\})} = \frac{\pi_{X,Y}(x, y)}{\alpha\pi_Y(y)} = \pi_X(x)$$

and

$$\begin{aligned}
\frac{P(x, y')}{P(\mathcal{X} \times \{y'\})} &= \frac{\alpha\pi_X(x) - \pi_{X,Y}(x, y)}{1 - \alpha\pi_Y(y)} \\
&= \pi_X(x)\frac{\alpha\pi_X(x) - \pi_{X,Y}(x, y)}{\pi_X(x) - \alpha\pi_X(x)\pi_Y(y)} \\
&= \pi_X(x)\frac{\alpha\pi_X(x) - \pi_{X,Y}(x, y)}{\pi_X(x) - \pi_{X,Y}(x, y)} \leq \pi_X(x).
\end{aligned}$$

For every $v \in \mathcal{Y}$ different from y and y', we have that $P(\mathcal{X} \times \{v\}) = 0$, so we may conclude from Lemma 4 that P satisfies (CI_2). The proof that P satisfies (CI_3) is completely symmetrical. □

4 The unimodal case

It turns out that when at least one of the marginal distributions π_X and π_Y is unimodal, the conditions for coherence under epistemic independence, stated in Theorem 8, simplify significantly: in this case, the necessary condition (NC) is also sufficient.

Theorem 10. *If the marginal distributions π_X and π_Y are not both plurimodal, then the normal joint distribution $\pi_{X,Y}$ is coherent under epistemic independence if and only if for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$:*

$$\pi_{X,Y}(x, y) \leq \frac{\pi_X(x)\pi_Y(y)\max\{\pi_X(x), \pi_Y(y)\}}{\pi_X(x) + \pi_Y(y) - \pi_X(x)\pi_Y(y)}.$$

Proof. It is enough to check that the condition is sufficient. Assume therefore that (NC) holds. It follows from Theorem 8 that $\pi_{X,Y}$ is coherent under epistemic independence if and only if for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$ there is some $P \in \mathcal{M}_i$ such that $P(x, y) = \pi_{X,Y}(x, y)$. This is what we now set out to prove. Let us assume without loss of generality that π_Y is unimodal with unique mode y_o, and let (x, y) be an arbitrary element of $\mathcal{X} \times \mathcal{Y}$. Lemma 9 (conditions 1 and 2) tells us that we may assume that $0 < \pi_{X,Y}(x,y)$ and $\max\{\pi_X(x), \pi_Y(y)\} < 1$. We show that in this case condition 3 of Lemma 9 holds because of the unimodality of π_Y, so that there is nothing left to prove. Indeed, there is some $x' \in \mathcal{X}$ such that $\pi_Y(y) = \pi_{X,Y}(x', y)$, whence we deduce that $\pi_X(x') = 1$ and therefore $x' \neq x$, using Lemma 3. Similarly, there is some $y' \in \mathcal{Y}$ such that $\pi_X(x) = \pi_{X,Y}(x, y')$, whence $\pi_Y(y') = 1$ and therefore $y' = y_o$, and $y' \neq y$. Also, there is some $y'' \in \mathcal{Y}$ such that $\pi_{X,Y}(x', y'') = \pi_X(x') = 1$, whence $\pi_Y(y'') = 1$, again by Lemma 3. Therefore $y'' = y' = y_o$, and $\pi_{X,Y}(x', y') = \pi_{X,Y}(x', y_o) = \pi_{X,Y}(x', y'') = 1$. $\square$

What we have in particular proved is that given two marginal possibility distributions π_X and π_Y, at least one of which is unimodal, the largest independent product possibility distribution that is coherent, is given by

$$\pi_{X,Y}(x,y) = T(\pi_X(x), \pi_Y(y)),$$

where T is the binary operator $T\colon [0,1]^2 \to [0,1]$ on the unit interval defined by

$$T(\alpha, \beta) = \frac{\alpha\beta \max\{\alpha, \beta\}}{\alpha + \beta - \alpha\beta}$$

for all α and β in $[0,1]$. The operator T is non-decreasing in both arguments, and has unit 1 and zero 0, so it is a so-called triangular seminorm. It is moreover continuous and commutative, but it is not a triangular norm, because it does not satisfy the associative property. To see this, take $\alpha = 1/4$, $\beta = 1/2$ and $\gamma = 3/4$; then $T(\alpha, T(\beta, \gamma)) = 81/1540 < 9/124 = T(T(\alpha, \beta), \gamma)$.

5 The general case

We now turn to the general case that both distributions π_X and π_Y may be plurimodal. The first thing to note is that the result of the previous case cannot be extended. To see this, consider the following counterexample.

Example 1. Let $\mathcal{X} = \{a_1, a_2, a_3\}$, $\mathcal{Y} = \{b_1, b_2, b_3\}$ and consider the normal joint possibility distribution $\pi_{X,Y}$ given by the following diagram:

$\pi_{X,Y}$	b_1	b_2	b_3	π_X
a_1	β	$\frac{3}{10}$	0	$\frac{3}{10}$
a_2	$\frac{1}{2}$	0	1	1
a_3	0	1	0	1
π_Y	$\frac{1}{2}$	1	1	

where, of course, $0 \leq \beta \leq 3/10$. It is clear that the necessary condition (NC) for coherence under independence is satisfied provided that $\beta \leq T(1/2, 3/10) = 3/26$. Assume that $\pi_{X,Y}$ is coherent under independence, which implies in particular that there is a $P \in \mathcal{M}_i$ such that $P(a_1, b_1) = \beta$, and which also implies that $\beta \leq 3/26$. Assume in addition that $\beta > 0$, whence $P(\{a_1\} \times \mathcal{Y}) > 0$ and $P(\mathcal{X} \times \{b_1\}) > 0$. There is some $\alpha \in (0,1)$ such that $\beta = \alpha\pi_X(a_1)\pi_Y(b_1)$ (use Lemma 3). Since $P(a_1, b_1)/P(\{a_1\} \times \mathcal{Y}) \leq \pi_Y(b_1)$ because $P \in \mathcal{M}_i$, it follows that $P(\{a_1\} \times \mathcal{Y}) \geq \alpha\pi_X(a_1)$, whence

$$P(a_1, b_2) \geq \alpha\pi_X(a_1) - \beta = \alpha\pi_X(a_1)[1 - \pi_Y(b_1)].$$

This implies that $P(\mathcal{X} \times \{b_2\}) > 0$. Consequently, it follows from $P(a_1, b_2)/P(\mathcal{X} \times \{b_2\}) \leq \pi_X(a_1)$ that

$$P(\mathcal{X} \times \{b_2\}) \geq \frac{P(a_1, b_2)}{\pi_X(a_1)} \geq \alpha[1 - \pi_Y(b_1)].$$

We find in a completely similar (or symmetrical) way that

$$P(\{a_2\} \times \mathcal{Y}) \geq \frac{P(a_2, b_1)}{\pi_Y(b_1)} \geq \alpha[1 - \pi_X(a_1)].$$

By combining these inequalities we find that

$$\begin{aligned}
&P(a_1,b_1)+P(a_2,b_1)+P(a_1,b_2)+P(a_2,b_3)+P(a_3,b_2) \\
&= P(a_1, b_1) + P(\{a_2\} \times \mathcal{Y}) + P(\mathcal{X} \times \{b_2\}) \\
&\geq \beta + \alpha[1 - \pi_Y(b_1)] + \alpha[1 - \pi_X(a_1)] \\
&= \beta\left(\frac{2}{\pi_X(a_1)\pi_Y(b_1)} - \frac{1}{\pi_X(a_1)} - \frac{1}{\pi_Y(b_1)} + 1\right) \\
&= \beta\frac{2 - \pi_X(a_1) - \pi_Y(b_1) + \pi_X(a_1)\pi_Y(b_1)}{\pi_X(a_1)\pi_Y(b_1)},
\end{aligned}$$

and if $\beta > 1/9$, or in other words, if

$$\pi_{X,Y}(a_1, b_1) > \frac{\pi_X(a_1)\pi_Y(b_1)}{2 - \pi_X(a_1) - \pi_Y(b_1) + \pi_X(a_1)\pi_Y(b_1)},$$

this contradicts the fact that P is a probability measure. We conclude that there can be no coherence for $\beta > 1/9$!

This counterexample gives us a hint about a sufficient condition for independence and coherence in the general case.

Theorem 11. *If the normal joint distribution $\pi_{X,Y}$ satisfies*

$$\pi_{X,Y}(x,y) \leq \min\Big\{T(\pi_X(x), \pi_Y(y)), \\ \frac{\pi_X(x)\pi_Y(y)}{2 - \pi_X(x) - \pi_Y(y) + \pi_X(x)\pi_Y(y)}\Big\}$$

for all $(x,y) \in \mathcal{X} \times \mathcal{Y}$, then it is coherent under epistemic independence.

Proof. Since (NC) is in particular satisfied, Theorem 8 tells us that we only have to show that for every $(x,y) \in \mathcal{X} \times \mathcal{Y}$ there is some $P \in \mathcal{M}_i$ such that $P(x,y) = \pi_{X,Y}(x,y)$. We infer from Lemma 9 (conditions 1 and 2) that we may assume that $0 < \pi_{X,Y}(x,y)$ and

$\max\{\pi_X(x), \pi_Y(y)\} < 1$. Then there are $x' \in \mathcal{X}$ and $y' \in \mathcal{Y}$ such that $\pi_{X,Y}(x', y) = \pi_Y(y)$ and $\pi_{X,Y}(x, y') = \pi_X(x)$. It follows from the assumptions and Lemma 3 that $\pi_X(x') = \pi_Y(y') = 1$, whence also $x' \neq x$ and $y' \neq y$. Lemma 9 (condition 3) tells us that we may assume that $\pi_{X,Y}(x', y') < 1$. Consequently, there are $x'' \neq x'$ in $\mathcal{X}$ and $y'' \neq y'$ in $\mathcal{Y}$ such that $\pi_{X,Y}(x', y'') = \pi_{X,Y}(x'', y') = 1$. Note that $\pi_X(x'') = \pi_Y(y'') = 1$, so π_X and π_Y are in this case plurimodal, $x'' \neq x$ and $y'' \neq y$. It also follows from the assumptions and Lemma 3 that there is some $\alpha \in (0,1)$ such that $\pi_{X,Y}(x,y) = \alpha\pi_X(x)\pi_Y(y)$. We can assume without loss of generality that $\pi_X(x) \leq \pi_Y(y)$. Let P be the probability measure uniquely defined on the power set of $\mathcal{X}\times\mathcal{Y}$ by $P(x,y) = \pi_{X,Y}(x,y)$,

$$\begin{aligned}
P(x', y) &= \alpha\pi_Y(y) - \pi_{X,Y}(x,y)\\
&= \pi_{X,Y}(x,y)\frac{1-\pi_X(x)}{\pi_X(x)}\\
P(x, y') &= \alpha\pi_X(x) - \pi_{X,Y}(x,y)\\
&= \pi_{X,Y}(x,y)\frac{1-\pi_Y(y)}{\pi_Y(y)}\\
P(x', y'') &= \alpha[1-\pi_X(x)-\pi_Y(y)] + \pi_{X,Y}(x,y)\\
&= \pi_{X,Y}(x,y)\frac{1-\pi_X(x)}{\pi_X(x)}\frac{1-\pi_Y(y)}{\pi_Y(y)}
\end{aligned}$$

and $P(x'', y') = 1-\alpha$. (It is easy to see that all these terms are non-negative and add up to one.) It only remains to be shown that $P \in \mathcal{M}_i$. We use Lemma 4. Recall that $\pi_{X,Y}(x,y) \leq \pi_{X,Y}(x,y') \leq \pi_{X,Y}(x',y) \leq \pi_{X,Y}(x',y'') = \pi_{X,Y}(x'',y') = 1$. Observe that $P(x,y) = \pi_{X,Y}(x,y)$ and that

$$P(x,y) + P(x,y') = \alpha\pi_X(x) < \pi_X(x) = \pi_{X,Y}(x,y').$$

Also $P(x,y) + P(x,y') + P(x',y)$ is equal to

$$\pi_{X,Y}(x,y)\frac{\pi_X(x)+\pi_Y(y)-\pi_X(x)\pi_Y(y)}{\pi_X(x)\pi_Y(y)}$$

and is therefore is dominated by $\pi_{X,Y}(x',y) = \pi_Y(y)$ if and only if

$$\pi_{X,Y}(x,y) \leq \frac{\pi_X(x)\pi_Y(y)^2}{\pi_X(x)+\pi_Y(y)-\pi_X(x)\pi_Y(y)},$$

which is implied by the hypothesis. We may therefore conclude from Lemma 4 that P satisfies (CI_1). Note also that $P(\{x\}\times\mathcal{Y}) = \alpha\pi_X(x) > 0$, $P(\{x'\}\times\mathcal{Y}) = \alpha[1-\pi_X(x)] > 0$ and $P(\{x''\}\times\mathcal{Y}) = P(x'',y') = 1-\alpha > 0$ and that $P(\{u\}\times\mathcal{Y}) = 0$ for all other $u \in \mathcal{X}$. Since $\pi_Y(y) \leq \pi_Y(y') = \pi_Y(y'') = 1$, $P(x'',y) = 0$,

$$\frac{P(x,y)}{P(\{x\}\times\mathcal{Y})} = \frac{\pi_{X,Y}(x,y)}{\alpha\pi_X(x)} = \pi_Y(y),$$

and

$$\frac{P(x',y)}{P(\{x'\}\times\mathcal{Y})} = \frac{\pi_{X,Y}(x,y)\dfrac{1-\pi_X(x)}{\pi_X(x)}}{\dfrac{\pi_{X,Y}(x,y)}{\pi_X(x)\pi_Y(y)}[1-\pi_X(x)]} = \pi_Y(y),$$

we infer from Lemma 4 that P satisfies (CI_3). Similarly, note that $P(\mathcal{X}\times\{y\}) = \alpha\pi_Y(y) > 0$, $P(\mathcal{X}\times\{y'\}) > 0$ and $P(\mathcal{X}\times\{y''\}) = P(x',y'') > 0$ and that $P(\mathcal{X}\times\{v\}) = 0$ for all other $v \in \mathcal{Y}$. Since $\pi_X(x) \leq \pi_X(x') = \pi_X(x'') = 1$, $P(x,y'') = 0$,

$$\frac{P(x,y)}{P(\mathcal{X}\times\{y\})} = \frac{\pi_{X,Y}(x,y)}{\alpha\pi_Y(y)} = \pi_X(x),$$

and since it is easily verified that $P(x,y')/P(\mathcal{X}\times\{y'\}) \leq \pi_X(x)$ if and only if

$$\pi_{X,Y}(x,y) \leq \frac{\pi_X(x)\pi_Y(y)}{2+\pi_X(x)\pi_Y(y)-\pi_X(x)-\pi_Y(y)}$$

which is implied by the hypothesis, we infer from Lemma 4 that P also satisfies (CI_2), so we may indeed conclude that $P \in \mathcal{M}_i$. □

This theorem provides us with a sufficient condition for the coherence under epistemic independence of possibility measures. The condition is not necessary, however. To see this, it is enough to consider the case that (NC) holds and one of the marginal distributions is unimodal, but where for some $(x,y) \in \mathcal{X}\times\mathcal{Y}$,

$$\begin{aligned}
\frac{\pi_X(x)\pi_Y(y)}{2-\pi_X(x)-\pi_Y(y)+\pi_X(x)\pi_Y(y)} &< \pi_{X,Y}(x,y)\\
&\leq \frac{\pi_X(x)\pi_Y(y)\max\{\pi_X(x),\pi_Y(y)\}}{\pi_X(x)+\pi_Y(y)-\pi_X(x)\pi_Y(y)}.
\end{aligned}$$

Then we deduce from Theorem 10 that $\pi_{X,Y}$ is coherent under epistemic independence. Still, $\pi_{X,Y}$ does not satisfy the condition given by the last theorem. The condition is not necessary in the case that both marginals are plurimodal either, as the following counterexample shows.

Example 2. Let $\mathcal{X} = \{a_1, a_2, a_3\}$, $\mathcal{Y} = \{b_1, b_2, b_3\}$ and consider the normal joint possibility distribution $\pi_{X,Y}$ given by the following diagram:

$\pi_{X,Y}$	b_1	b_2	b_3	π_X
a_1	β	$\frac{3}{10}$	0	$\frac{3}{10}$
a_2	$\frac{1}{2}$	1	1	1
a_3	0	1	0	1
π_Y	$\frac{1}{2}$	1	1	

where $1/9 < \beta < 3/26$. Then $\pi_{X,Y}(a_1,b_1) = \beta$ does not satisfy the condition stated on the previous theorem, as

$$\frac{\pi_X(a_1)\pi_Y(b_1)}{2-\pi_X(a_1)-\pi_Y(b_1)+\pi_X(a_1)\pi_Y(b_1)} = \frac{1}{9}.$$

We show that $\pi_{X,Y}$ is nevertheless coherent under independence. Clearly, (NC) is satisfied, as $\beta < 3/26 = T(1/2, 3/10)$. Consider $(x, y) \in \mathcal{X} \times \mathcal{Y}$, then we show that there is a $P \in \mathcal{M}_i$ such that $P(x, y) = \pi_{X,Y}(x, y)$. It follows from Lemma 9 (conditions 1 and 2) that we may assume that $\pi_{X,Y}(x, y) > 0$ and $\max\{\pi_X(x), \pi_Y(y)\} < 1$, so we need only look at $x = a_1$ and $y = b_1$. Note that $\pi_{X,Y}(a_1, b_2) = \pi_X(a_1)$, $\pi_{X,Y}(a_2, b_1) = \pi_Y(b_1)$ and $\pi_{X,Y}(a_2, b_2) = 1$ so Lemma 9 (condition 3) tells us that there is a $P \in \mathcal{M}_i$ such that $P(a_1, b_1) = \pi_{X,Y}(a_1, b_1) = \beta$, and $\pi_{X,Y}$ is coherent under independence.

6 Conclusions

In this paper, we have continued the study of the implications of giving possibility measures a behavioural interpretation in terms of upper betting rates, initiated in [3, 4, 5, 14, 15]. In particular, we have looked at the consequences of the rationality requirements of avoiding sure loss and coherence when forming independent products of marginal possibility measures. The definition of independence that was used here, is based on Walley's [13] notion of epistemic independence: two variables are epistemically independent for a subject when his beliefs about the value taken by one variable are not influenced by new knowledge about the value of the other variable. In the context of possibility theory, where beliefs are expressed in terms of possibility measures, it seems natural to express epistemic independence in terms of the equality of conditional and marginal possibility distributions (or measures), as we did in Definition 1. We have obtained a simple characterisation for the coherence under independence of a joint possibility distribution in the unimodal case, and we have found a simple sufficient condition, as well as a different, necessary one in the plurimodal case. It is not clear to us whether in the general case, there is a simple necessary and sufficient condition involving only the *local* values of the joint and marginal possibility distributions.

An immediate conclusion of Lemma 3 and Theorems 10 and 11 is that the so-called minimum and product rules for forming joint distributions from given marginals, which yield $\pi_{X,Y}(x, y) = \min\{\pi_X(x), \pi_Y(y)\}$ and $\pi_{X,Y}(x, y) = \pi_X(x)\pi_Y(y)$ respectively, and which are quite common in possibility theory (see for instance [2, 6, 7, 17]), are only coherent when $\pi_X(x)$ and $\pi_Y(y)$ assume only the values 0 and 1.

We could also consider the so-called independent natural extension $\overline{E}$ [13, Section 9.3] of two marginal possibility measures Π_X and Π_Y. This is the greatest (least-committal or most conservative) coherent and independent joint *upper probability*, which need not be a possibility measure. In fact, on products $A \times B$ it can be shown that $\overline{E}(A \times B) = \Pi_X(A)\Pi_Y(B)$, where $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$ [13, Section 9.3.5]. $\overline{E}$ will therefore in general *not* be a possibility measure: if it were, its distribution would be given by the product rule, which is generally not coherent!

The results in this paper indicate that the theory of imprecise probabilities has useful things to say about independence in possibility theory. But we should warn the reader against too much optimism. Indeed, possibility measures are rather imprecise uncertainty models: if Π is a normal possibility measure (and therefore a coherent upper probability) on some set Ω, and N is its conjugate lower probability, also called necessity measure, and defined by $\mathrm{N}(A) = 1 - \Pi(\mathrm{co}A)$, where $\mathrm{co}A$ is the set-theoretic complement of $A \subseteq \Omega$, then we have that $\Pi(A) < 1 \Rightarrow \mathrm{N}(A) = 0$: the probability interval $[\mathrm{N}(A), \Pi(A)]$ always contains zero or one (or both). Alternatively, it always holds for $A \subseteq \Omega$ that $\Pi(A) = 1$ or $\Pi(\mathrm{co}A) = 1$, meaning that a subject whose beliefs are modelled by the upper probability Π will not be disposed to bet against A or against $\mathrm{co}A$, and this for all $A \subseteq \Omega$. On a behavioural interpretation, possibility measures therefore model fairly weak information states. On the other hand, a judgement of independence is quite informative, and we suspect that in some cases it will be too informative to be adequately modelled by possibility measures, or within the context of possibility theory. This is illustrated by the fact that, as we have seen above, the greatest independent joint possibility measure $T(\Pi_X(A), \Pi_Y(B))$ can be appreciably smaller than the independent natural extension $\overline{E}(A \times B) = \Pi_X(A)\Pi_Y(B)$ on products $A \times B$: if we restrict ourselves to possibilistic models, we are obliged, in order to capture independence, to use products that are more precise than if we had used a more general approach, e.g., with coherent upper probabilities. This identifies a weakness in possibility theory.

We also want to warn the reader against too careless an interpretation of our results. To see what is involved here, let us consider the following very simple example.

Example 3. Let X take values in $\mathcal{X} = \{a_1, a_2\}$ and let Y take values in $\mathcal{Y} = \{b_1, b_2\}$. Assume that we know that X and Y jointly can only assume the values (a_1, b_1) or (a_2, b_2), and nothing more. This is clearly incompatible with the epistemic independence of X and Y: if we know what value Y takes, we know the value of X, and vice versa. It is often argued that the given information can be modelled by a joint possibility distribution $\pi_{X,Y}$, with

$$\begin{aligned} \pi_{X,Y}(a_1, b_1) &= \pi_{X,Y}(a_2, b_2) = 1 \\ \pi_{X,Y}(a_1, b_2) &= \pi_{X,Y}(a_2, b_1) = 0. \end{aligned} \tag{3}$$

This is equivalent to observing that we are prepared to bet at any odds against the event $\{(a_1, b_2), (a_2, b_1)\}$. The lower probability (or necessity) $N_{X,Y}(\{(a_k, b_k)\})$ is zero, which models that we are not prepared to bet on the occurrence of $\{(a_k, b_k)\}$ at any odds, $k = 1, 2$. This is reasonable, because we have *no information at all* about which of the two events $\{(a_1, b_1)\}$ and $\{(a_2, b_2)\}$ will occur.

Not surprisingly, the marginal distributions are vacuous, or completely uninformative: $\pi_X(a_1) = \pi_X(a_2) = 1$ and $\pi_Y(b_1) = \pi_Y(b_2) = 1$. What may seem suprising, however, is that, according to Theorem 8 and Lemma 9, the joint distribution $\pi_{X,Y}$ *is coherent under epistemic independence*: $\pi_{X,Y}$ is a perfectly rational *independent* product of the marginals π_X and π_Y, even if it is not the most conservative one! This seems to contradict our earlier observation that the available knowledge is incompatible with the epistemic independence of X and Y.

To see what goes wrong, we need to look at the conditional possibility distributions $\{\pi_{Y|X}(\cdot|x) \colon x \in \mathcal{X}\}$ and $\{\pi_{X|Y}(\cdot|y) \colon y \in \mathcal{Y}\}$. It follows from Theorem 1 that they are coherent with $\pi_{X,Y}$ if and only if

$$\pi_{X|Y}(a_k|b_k) = \pi_{Y|X}(b_k|a_k) = 1, \quad k = 1, 2.$$

This means that, in particular, both the vacuous conditional distributions

$$\pi_{X|Y}(a_k|b_\ell) = \pi_{Y|X}(b_\ell|a_k) = 1, \quad k, \ell = 1, 2 \tag{4}$$

and the precise conditional distributions

$$\pi_{X|Y}(a_k|b_\ell) = \pi_{Y|X}(b_\ell|a_k) = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases} \tag{5}$$

are coherent with the joint distribution $\pi_{X,Y}$: contrary to what we are often used to in (precise) probability theory, we cannot rely on coherence to provide us with unique conditional distributions (see also [15]). The joint distribution (3) *is not a fully adequate model of the given information!* To remedy this, we have to repress an ingrained (precise) probabilistic reflex and also specify the conditional distributions $\{\pi_{Y|X}(\cdot|x) \colon x \in \mathcal{X}\}$ and $\{\pi_{X|Y}(\cdot|y) \colon y \in \mathcal{Y}\}$ explicitly, as these cannot be determined uniquely from the joint $\pi_{X,Y}$. Now the only conditional distributions that reflect the given information are given by (5). Of course, these are different from the marginal distributions, reflecting that the variables X and Y are not epistemically independent. The problem above could only occur because we assumed that the joint distribution adequately represents the given information. But it doesn't: as far as coherence is concerned, this joint is compatible with the vacuous conditional distributions (4), for which the epistemic independence conditions (2) are satisfied.

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