Coherent Risk Measures and Upper Previsions

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Abstract

In this paper coherent risk measures and other currently used risk measures, notably Value-at-Risk (VaR), are studied from the perspective of the theory of coherent imprecise previsions. We introduce the notion of coherent risk measure defined on an arbitrary set of risks, showing that it can be considered a special case of coherent upper prevision. We also prove that our definition generalizes the notion of coherence for risk measures defined on a linear space of random numbers, given in literature. We also show that Value-at-Risk does not necessarily satisfy a weaker notion of coherence called 'avoiding sure loss' (ASL), and discuss both sufficient conditions for VaR to avoid sure loss and ways of modifying VaR into a coherent risk measure.

Keywords. Coherent risk measure, imprecise prevision, Value-at-Risk, avoiding sure loss condition.

1 Introduction

The notion of coherent risk measure has been introduced very recently in a series of papers, including [1, 2, 4], by P. Artzner, F. Delbaen, S. Eber and D. Heath. A coherent risk measure is defined through a set of axioms on a linear space of random numbers. These axioms are quite reasonable when trying to give a numerical evaluation to risks, but are not all necessarily satisfied by most currently used risk measures, including Value-at-Risk (VaR).

The theory of coherent risk measures is related to various other theories. In particular, it is noted in [4] that the theory of imprecise probabilities 'is certainly not disjoint from risk management considerations'.

In this paper we introduce the notion of coherent risk measure defined on arbitrary sets of risks and show that this notion is strictly related to the notion of imprecise prevision. More generally, the paper is also concerned with other risk measures, notably VaR, from the perspective of imprecise previsions.

The theory of imprecise previsions has been extensively studied by P. Walley [9]; previous work on the topic includes [12] and the pioneering paper [7]. The theory generalizes de Finetti's approach to precise previsions [3], is extremely general and flexible and includes also other uncertainty measures as special cases, for instance belief functions or 2-monotone lower probabilities. We show in this paper that it can also be applied to risk measures.

To be more precise, after introducing in Section 2.1 a behavioural interpretation of risk measures, we show that this interpretation leads us to consider them a special case of upper previsions, whose basic properties are shortly reported in Section 2.2. This fact lets us apply to risk measures, in Section 2.3, the well-known consistency notions of *avoiding sure loss* (ASL) and *coherence* from the theory of imprecise previsions. In this way, the notion of coherent risk measure is generalized, being defined for *arbitrary* (finite or not, structured or not) sets of risks in Definition 4. It is shown in Section 2.4 that this definition extends the definition of coherence for risk measures given in literature. Besides, using the theory of imprecise previsions, known properties of coherent risk measures defined on linear spaces can be easily obtained in this more general setting (we quote the use of 'scenarios' to obtain coherent risk measures, see Section 2.5). We prove in Section 3.1 that VaR, which is known not to be coherent, does not even necessarily satisfy the weaker consistency condition of avoiding sure loss. However, it can be seen in a fairly general case that VaR tends to avoid sure loss if the VaR evaluation is 'sufficiently prudential': see Section 3.1, especially Proposition 4, Corollary 1 and Subsection 3.1.1.

If a risk measure avoids sure loss there is a standard way to correct it into a coherent risk measure, resorting to the notion of natural extension [9] of imprecise previsions, as discussed in Section 3.2.

2 Coherent risk measures as upper previsions

2.1 Behavioural interpretation of risk measures

In this section we present a behavioural interpretation of risk measures which will lead us to a general definition of the notion of coherent risk measure in Section 2.3. In addition, we explain some assumptions made to settle the problem and to ensure comparability with previous work on the subject.

Risk measures are naturally introduced when considering the problem of evaluating at time $t = 0$ how risky the positions in a set $\mathcal{D}$ are, based on the future (random) value of each position X at a fixed time $T > 0$. Technically, every X is a real random number, which is given the meaning of the value at T of a certain position. We shall sometimes call X a *risk* as done e.g. in [2] (anyway, there is no general agreement in financial literature on the usage of the term risk). We assume throughout the paper that every risk in $\mathcal{D}$ is bounded.

Given $\mathcal{D}$, a risk measure ρ is a mapping from $\mathcal{D}$ into $I\!R$ assigning to each $X \in \mathcal{D}$ a number, which is a risk assessment for X. From a behavioural point of view, a subject willing to evaluate $\rho(X)$ can identify it with the infimum of the amounts that he/she would ask (at time $t = 0$) to shoulder the risk X (at time T). Clearly, the more X is risky the higher $\rho(X)$ should be. Since getting a specific amount for receiving X is the same as selling $-X$ for the same amount, $\rho(X)$ can be equivalently viewed as an infimum selling price (at 0) for receiving $-X$ (at T). Note that in this scheme there is a gap between time 0 when ρ is assessed and time T when the value of X is determined. This fact cannot usually be neglected: cashing an amount today is not financially equivalent to receiving the same amount at an arbitrary future time T. Therefore, the future position X at T has to be discounted to determine its present value at 0 and make it financially homogeneous with $\rho(X)$. We shall assume that

a) there exists a reference instrument in the market which yields a 'sure' amount $r > 0$ at $t = T$ for every monetary unit invested (at $t = 0$).

Generally, such an instrument can be a zero-coupon bond with maturity at T, while $r = 1 + i(T)$ and $i(T)$ is the interest accumulated over the period $[0, T]$ by investing one monetary unit at $t = 0$ (estimating this interest as a function of T is a standard problem in the mathematics of finance, see for instance [5]).

We can therefore refer the behavioural interpretation of $\rho(X)$ to the present value of X (or $-X$), i.e. to X/r. This is not necessary when r is sufficiently close to one or the time gap is not significantly large. These assumptions are sometimes understood, see e.g. [9] Section 2.4.8.

Summing up, the risk measure for X can be viewed as the infimum selling price for $-X/r$.

This is precisely the behavioural interpretation given in [9] for the *upper prevision* $\overline{P}$ of $-X/r$. Therefore, a risk measure appears to be a special case of upper prevision:

$$\rho(X) = \overline{P}(-X/r) \tag{1}$$

The sign of ρ lets us distinguish between desirable and non-desirable risks. In fact, $\rho(X) < 0$ means that the subject would be disposed to receive a negative sum, i.e. to pay something, for getting X at T. In other words, X is desirable to him/her. On the contrary, $\rho(X) > 0$ implies that the subject should be actually paid to undergo the risk X. The case $\rho(X) = 0$ identifies a marginally desirable risk (i.e. $X + \varepsilon$ is desirable for every $\varepsilon > 0$). Hence, the behavioural interpretation of ρ implies that:

i) X is desirable if $\rho(X) \leq 0$

ii) X is not desirable if $\rho(X) > 0$.

2.2 Consistency of imprecise previsions

Relation (1) highlights the strict correspondence between the behavioural interpretation of risk measures and upper previsions. It appears therefore natural to apply to risk measures the same consistency notions developed for upper previsions in the theory of coherent imprecise previsions, as well as the main results of this theory. The basic notions which will be needed later on are shortly recalled in this section. Proofs of the results reported here and more detailed information may be found in [9] (see also [10] for a shorter introduction to the topic). A first coherence condition is given by the following definition [9].

Definition 1 *Given a set $\mathcal{D}$ of (bounded) random numbers, a mapping $\overline{P}$ from $\mathcal{D}$ into $I\!R$ is an upper prevision on $\mathcal{D}$ that* avoids sure loss *iff, for all $n \in \mathbf{N}$, for each $X_1, \ldots, X_n \in \mathcal{D}$, for each $s_1, \ldots, s_n \geq 0$, it is*

$$\sup \sum_{i=1}^{n} s_i(\overline{P}(X_i) - X_i) \geq 0$$

The condition of avoiding sure loss (ASL) appears to be too weak in many respects. For instance, it may be compatible with the inequality $\overline{P}(X) > \sup X$ or it may violate monotonicity. Therefore, it is usual to require for a prevision $\overline{P}$ to satisfy a stronger condition of coherence [9].

Definition 2 *Given an arbitrary set $\mathcal{D}$ of (bounded) random numbers, a mapping $\overline{P}$ from $\mathcal{D}$ into $I\!R$ is a* coherent upper prevision *for the random numbers in $\mathcal{D}$ iff, for all $n \in \mathbf{N}$, for each $X_0, X_1, \ldots, X_n \in \mathcal{D}$, for each $s_0, s_1, \ldots, s_n$ real and non-negative, defining $\overline{G} = \sum_{i=1}^{n} s_i(\overline{P}(X_i) - X_i) - s_0(\overline{P}(X_0) - X_0)$, it is* $\sup \overline{G} \geq 0$.

This definition generalizes to imprecise previsions the *coherence principle* introduced by de Finetti [3] for (precise) previsions [1] and probabilities. It also includes the definition of coherent upper probability, which is obtained when each $X \in \mathcal{D}$ is the *indicator function* $|E|$ of some event E ($|E|$ is the random number which is equal to 1 if E is true, to 0 if E is false).

The theory of coherent imprecise previsions is very general, as shown by its following basic features:

i) A coherent precise or imprecise prevision for a random number X does *not* require any prior assignment of a (precise or imprecise) probability distribution on X, and in this sense a prevision is a more flexible tool than an expectation, since it fits well with situations where first-order moment evaluations only are available. However, whenever an expectation is defined, it is a coherent prevision, so the two concepts have the same meaning, in the precise case, of summarizing X. Imprecision in evaluation - whatever are its sources - brings to consider an upper and a lower prevision. Anyway, it is sufficient to refer to either upper ($\overline{P}$) or lower ($\underline{P}$) previsions, assuming that

$$\underline{P}(X) = -\overline{P}(-X) \tag{2}$$

holds.

We shall mainly employ upper previsions, since this choice is more natural to interpret coherent risk measures as imprecise previsions.

ii) Coherent upper previsions are defined on *quite arbitrary* (non-empty) sets of random numbers, and can therefore be applied in very general (and common) situations. Clearly, the results we mention for imprecise previsions (like Theorem 2 below) also hold in the same general setting.

iii) Given a coherent upper prevision $\overline{P}$ on a set $\mathcal{D}$, there *always* exists a coherent extension $\overline{P}'$ (which is generally not unique) on *any* superset $\mathcal{D}' \supset \mathcal{D}$, i.e. $\overline{P}'$ is coherent on $\mathcal{D}'$ and equal to $\overline{P}$ on $\mathcal{D}$.

Although sometimes too weak, the condition of avoiding sure loss is anyway relevant to the theory of imprecise previsions because:

1) it is easier to assess (and to check) than coherence;

2) given an upper prevision $\overline{P}$ on $\mathcal{D}$ that avoids sure loss there *always* exists a canonical way of correcting $\overline{P}$ into a coherent prevision $\overline{P}_E$, which can be defined on any set of random numbers $\mathcal{D}' \supset \mathcal{D}$. $\overline{P}_E$ is termed *natural extension* in [9] and is a key concept in the theory of coherent upper previsions. We shall discuss it in Section 3.2.

Let us define $\mathcal{M}(\overline{P})$ as the set of all precise coherent previsions P on $\mathcal{D}$ which are *dominated* by $\overline{P}$ on $\mathcal{D}$, i.e. are such that $P(X) \leq \overline{P}(X)$ for each $X \in \mathcal{D}$. Upper previsions that avoid sure loss and coherent upper previsions may be characterized as follows [9]:

Theorem 1 *An upper prevision $\overline{P}$ on $\mathcal{D}$ avoids sure loss iff $\mathcal{M}(\overline{P}) \neq \emptyset$.*

Theorem 2 ***(Upper envelope theorem)*** *An upper prevision $\overline{P}$ on $\mathcal{D}$ is coherent iff ($\mathcal{M}(\overline{P}) \neq \emptyset$ and)*

$$\overline{P}(X) = \sup\left\{P(X) : P \in \mathcal{M}(\overline{P})\right\} \tag{3}$$

for all $X \in \mathcal{D}$ (actually, sup *is attained).*

We shall discuss in Section 2.5 some relevant implications of the upper envelope theorem in our context.

Several necessary conditions for coherence are known [9], but no subset of such conditions seems to be also sufficient for coherence when $\mathcal{D}$ is arbitrary. The following necessary conditions will be used in the sequel:

a) $\overline{P}(X + \alpha) = \overline{P}(X) + \alpha$, $\forall\, \alpha \in I\!R$;

b) if $X \leq Y$ then $\overline{P}(X) \leq \overline{P}(Y)$ (monotonicity).

However, coherent upper previsions may be characterized through some simple axioms if $\mathcal{D}$ has a special structure. The most relevant case is the following:

Theorem 3 *Let $\mathcal{L}$ be a linear space and $\overline{P}$ a mapping from $\mathcal{L}$ into $I\!R$. Then $\overline{P}$ is coherent on $\mathcal{L}$ iff:*

[1] The definition of coherent precise prevision may be obtained from Definition 2 by simply dropping the non-negativity restriction for $s_0, s_1, \ldots, s_n$.

P1) $\overline{P}(X) \le \sup X, \forall X \in \mathcal{L}$

P2) $\overline{P}(\lambda X) = \lambda \overline{P}(X), \forall X \in \mathcal{L}, \forall \lambda > 0$

S) $\overline{P}(X+Y) \le \overline{P}(X) + \overline{P}(Y), \forall X, Y \in \mathcal{L}$.

2.3 Consistency of risk measures

Relation (1) and the avoiding sure loss and coherence definitions in Section 2.2 let us define in a natural way two corresponding consistency definitions for risk measures defined on arbitrary sets of random numbers.

Definition 3 *Given an arbitrary set $\mathcal{D}$ of random numbers, a mapping ρ from $\mathcal{D}$ into $I\!R$ is a risk measure that* avoids sure loss *iff there exists an upper prevision $\overline{P}$ on $\mathcal{D}^* = \{-X/r : X \in \mathcal{D}\}$ that avoids sure loss such that $\rho(X) = \overline{P}(-X/r), \forall X \in \mathcal{D}$.*

Definition 4 *Given an arbitrary set $\mathcal{D}$ of random numbers, a mapping ρ from $\mathcal{D}$ into $I\!R$ is a* coherent risk measure *on $\mathcal{D}$ iff there exists a coherent upper prevision $\overline{P}$ defined on $\mathcal{D}^* = \{-X/r : X \in \mathcal{D}\}$ such that $\rho(X) = \overline{P}(-X/r)$.*

Clearly, the behavioural justification of these conditions is strictly related to the behavioural justification of the corresponding conditions for upper previsions (see [9] for a wide discussion). For instance, if ρ incurs sure loss on $\mathcal{D}$, a person who considers $\rho(X)$ as the infimum of the amounts he would charge to accept X can be made a sure loser by offering him $X_1, \ldots, X_n \in \mathcal{D}$ together with some amounts $\mu_i > \rho(X_i)$ $(i = 1, \ldots n)$, for some choice of the non-negative coefficients $s_1, \ldots, s_n$ [2].

Practical risk evaluations on each position in $\mathcal{D}$ are often made by a so-called *regulator*. The regulator does not necessarily own or manage the positions, but might be jointly liable in cases of insolvency or unsatisfactory results and often has the power of deciding whether a risk can be undertaken or not (for instance, when the regulator is a government agency controlling insurance activity or a holding controlling some of its subsidiaries). So it is the regulator who could be made a sure loser in the hypotheses just described above. There is anyway a difference between this situation and other common applications of the theory of imprecise previsions like, say, bookmaking: here the counterparts - the regulator and the insurance company, for instance - usually cooperate to avoid sure losses (if the regulator bears a sure loss, that's because the insurance company is insolvent), whilst the bookie would very much like his opponent(s) to suffer from a sure loss (and vice versa).

[2] The coefficients can always be (non-negative) integer numbers, because the condition of avoiding sure loss can be equivalently defined replacing $s_1, \ldots, s_n \ge 0$ with $s_1, \ldots, s_n \in \mathrm{N}$ and allowing $X_1, \ldots, X_n$ to be not necessarily distinct (the same applies to the coherence definition). This lets every X_i represent non infinitely divisible quantities.

If a risk measure ρ is coherent, it can be given an interesting interpretation as follows. Consider the risk $X + k$, where k is a real constant. By definition and property a) in Section 2.2,

$$\rho(X+k) = \overline{P}(-(X+k)/r) = \overline{P}(-X/r) - k/r$$
$$= \rho(X) - k/r.$$

From this and i) in Section 2.1 it follows that, when $\rho(X) > 0$, $\rho(X)$ is the present value at time $t = 0$ of the minimum amount k which must be added to the risk X at T to make the risk $X + k$ desirable. Operationally, the simplest way to add this minimum amount k to X at T is to make an investment of $\rho(X)$ in the reference instrument at $t = 0$ $(k = r\rho(X))$ Analogously, if $\rho(X) < 0$, $-\rho(X)$ is the present value at $t = 0$ of the maximum amount h which can be subtracted from X at T, keeping $X - h$ desirable.

2.4 Coherent risk measures on linear spaces

Coherent risk measures have been recently introduced in a series of papers, including [1, 2, 4], by P. Artzner, F. Delbaen, S. Eber and D. Heath. They have been defined through a set of axioms on a linear space of random numbers. In this section we show that our concept of coherent risk measure generalizes the one given in literature and reduces to it when the extra assumption is made that the set of risks $\mathcal{D}$ under evaluation is a linear space. Clearly, this assumption is fairly restrictive and not quite realistic in practice. Consider also that it is guaranteed from iii) of Section 2.2 that whenever we need to evaluate risks in a larger set $\mathcal{D}'$, any coherent risk measure on $\mathcal{D}$ can be coherently extended to $\mathcal{D}'$. So it is really unnecessary to define coherent risk measures on structured sets of random numbers only. Nevertheless, when no structure is required, it does not seem possible to characterize coherence of previsions through a simple system of axioms (see Section 2.7 in [9]). This suggests that sets of axioms including subadditivity and other important and also intuitively necessary conditions for coherence of risk measures are not sufficient to characterize these measures, unless convenient constraints are imposed on $\mathcal{D}$.

We shall refer to [2] to recall the basic assumptions and the definition of coherent risk measure on linear spaces. Although other papers, like [1] and [4], modify some of these assumptions, we shall not consider these variants here, since they would leave essentially unchanged the conclusions of this paper.

It is assumed in [2] that:

a) hypothesis a) of Section 2.1 holds;
b) the set of random numbers $\mathcal{D}$ is a linear space, $\mathcal{D} = \mathcal{L}$. [3]

Also, in [2] a fixed finite partition describes all distinct values of every X in $\mathcal{L}$ and, consequently, every X is a *simple* random number, i.e. can have a finite number of distinct values. Anyway, the finiteness assumption is unnecessary in our framework and will not be made here. Given a) and b), a coherent risk measure is defined in [2] as follows:

Definition 5 *Let $\mathcal{L}$ be a linear space of random numbers. A mapping ρ from $\mathcal{L}$ into $I\!R$ is a* coherent risk measure *iff it satisfies the following axioms:*

T) $\forall\, X \in \mathcal{L}, \forall\, \alpha \in I\!R$, $\rho(X+\alpha r) = \rho(X) - \alpha$ *(translation invariance)*

PH) $\forall\, X \in \mathcal{L}, \forall\, \lambda \geq 0$, $\rho(\lambda X) = \lambda\rho(X)$ *(positive homogeneity)*

M) $\forall\, X, Y \in \mathcal{L}$, *if* $X \leq Y$ *then* $\rho(Y) \leq \rho(X)$ *(monotonicity)*

S) $\forall\, X, Y \in \mathcal{L}$, $\rho(X+Y) \leq \rho(X) + \rho(Y)$ *(subadditivity)*

In order to make use of Theorem 3 in a convenient form for our purpose, we formulate the following lemma, whose trivial proof is omitted.

Lemma 1 *Let $\mathcal{L}$ be a linear space of random numbers, r a positive real number. A mapping μ from $\mathcal{L}$ into $I\!R$ satisfies the following axioms:*

P1) $\mu(X) \leq \sup(-X/r)$, $\forall\, X \in \mathcal{L}$

P2) $\mu(\lambda X) = \lambda\mu(X)$, $\forall\, X \in \mathcal{L}$, $\forall\, \lambda > 0$

S) $\mu(X+Y) \leq \mu(X) + \mu(Y)$, $\forall\, X, Y \in \mathcal{L}$

if and only if there exists a coherent upper prevision $\overline{P}$ on $\mathcal{L}$ such that $\mu(X) = \overline{P}(-X/r)$.

Then we have:

Proposition 1 *Let $\mathcal{L}$ be a linear space of random numbers. A mapping ρ from $\mathcal{L}$ into $I\!R$ satisfies axioms T), PH), M) and S) of Definition 5 if and only if it satisfies axioms P1), P2), S) of Lemma 1.*

[3] The model in [2] considers also investments in possibly different countries and therefore includes future random exchange rates. To simplify the sequel, we shall not include this variant. It any case, it can be easily introduced without substantial modifications in the results.

Proof Let us prove that T), PH), M) and S) imply P1), P2), S). Actually, the only non-trivial implication is to see that P1) holds, which can be shown applying subsequently M) and T) (noting that PH) implies $\rho(0) = 0$), starting from the inequality $\inf X \leq X$ and obtaining

$$\rho(X) \leq \rho(\inf X) = \rho(\frac{\inf X}{r} r)$$
$$= -\inf X/r = \sup\,(-X/r)\,.$$

Assume now that P1), P2), S) hold. We only have to show that T), PH) with $\lambda = 0$, and M) hold.
To obtain T), note that if ρ satisfies P1), P2), S) then $\rho(X) = \overline{P}(-X/r)$ by Lemma 1. Applying the necessary condition for coherence a) in Section 2.2,

$$\rho(X+\alpha r) = \overline{P}(-(X+\alpha r)/r)$$
$$= \overline{P}(-X/r) - \alpha = \rho(X) - \alpha.$$

Use P1) with $X = 0$ to obtain $\rho(0) \leq 0$. On the other hand, from S) $\rho(0) = \rho(0+0) \leq \rho(0) + \rho(0)$, so that $\rho(0) \geq 0$. Hence $\rho(0) = 0$ and PH) holds also for $\lambda = 0$.
Finally, suppose $X \leq Y$. From Lemma 1 and condition b) in Section 2.2 it follows

$$\rho(Y) = \overline{P}(-Y/r) \leq \overline{P}(-X/r) = \rho(X)$$

so that M) holds too. ■

Applying Lemma 1 and Proposition 1, and recalling Definition 5, we immediately obtain the following basic result:

Proposition 2 *Let $\mathcal{L}$ be a linear space of random numbers. A mapping ρ from $\mathcal{L}$ into $I\!R$ is a coherent risk measure according to Definition 5 if and only if there exists a coherent upper prevision $\overline{P}$ on $\mathcal{L}$ such that $\rho(X) = \overline{P}(-X/r)$, for every $X \in \mathcal{L}$.*

We can therefore conclude that Definition 4 and Definition 5 coincide when the set of risks $\mathcal{D}$ is endowed with a linear space structure.

2.5 Obtaining coherent risk measures via envelope theorems

Coherent risk measures do not necessarily require the knowledge of an underlying probability distribution for each X, and with respect to this they differ from other risk measures, like VaR to be discussed in the next section (see also the risk measure in 3.1.2). This fact does not seem to have been recognized in earlier literature on coherent risk measures, as appears from prior use of upper envelope theorems to assess ρ. In fact, in our framework the following proposition holds, ensuing from the upper envelope theorem and Definition 4.

Proposition 3 *ρ is a coherent risk measure on a set $\mathcal{D}$ if and only if*

$$\rho(X) = \sup\{P(-X/r) : P \in \mathcal{P}\} \tag{4}$$

where $\mathcal{P}$ is a (non-empty) set of coherent precise previsions on $\mathcal{D}^* = \{-X/r : X \in \mathcal{D}\}$.

Proposition 3 justifies the correctness of procedures for getting a risk measure out of 'scenarios', where a scenario is a (precise) coherent prevision P on $\mathcal{D}^*$ and the measure is obtained for each $X \in \mathcal{D}$ performing the supremum of $P(-X/r)$ over all considered scenarios.

As hinted above, the use of envelope theorems with these purposes is not new in risk measure theory (see [2]), but Proposition 3 exploits the basic features i) and ii), Section 2.2, of coherent imprecise previsions to achieve a more general usage of scenarios than it was previously recognized. Precisely, from ii) a scenario may concern only the risks we are really interested in, i.e. those in $\mathcal{D}$, and, for each $X \in \mathcal{D}$, by i) a scenario requires only assessing a precise prevision for X, a much simpler task than evaluating its probability distribution.

3 VaR and coherence

Some risk measures which are not coherent are commonly used in practice, like Value-at-Risk (VaR). VaR is not coherent, since it is not subadditive [2].

It is an interesting problem to see how far from coherence are incoherent risk measures. To provide some answers, we discuss the relationship between the avoiding sure loss condition (Definition 3) and VaR in Section 3.1, and ways of 'correcting' VaR and, more generally, other incoherent risk measures in Section 3.2.

3.1 VaR does not necessarily avoid sure loss

We shall use the following definition of VaR_α, taken from [2]:

Definition 6 *Let X be a random number, whose probability distribution is P. The number q is an α-*quantile *for X if*

$$P(X < q) \leq \alpha \leq P(X \leq q) \tag{5}$$

Define then

$$q_\alpha^+(X) = \inf\{x : P(X \leq x) > \alpha\} \tag{6}$$

$$VaR_\alpha(X) = -q_\alpha^+(X/r) \tag{7}$$

Clearly, given P, we obtain more prudent risk evaluations from VaR_α when α gets smaller.

In order to be assigned on a set $\mathcal{D}$ of random numbers, VaR requires more (and more precise) information than a coherent risk measure: we should be able to assess (at least) the marginal probability distribution P for each $X \in \mathcal{D}$, instead of giving (imprecise) first-order moment evaluations.

The following example shows that VaR may fail to avoid sure loss.

Example Let $I\!P$ be a finite partition, whose *atoms* (not impossible events, mutually disjoint and whose logical sum is the sure event Ω) are $e_1, \ldots, e_n$.

Denoting with $|e_i|$ the indicator function of e_i, let $\mathcal{D} = \{X_i : X_i = -|e_i|, i = 1, \ldots, n\}$, and assign a probability distribution on $I\!P$, such that $P(X_i = -1) = P(e_i) = p_i, i = 1, \ldots, n$.

We prove now that VaR_α *incurs sure loss on $\mathcal{D}$ if and only if $\alpha \geq \max\{p_1, \ldots, p_n\}$.*

Preliminarily, consider $X_i \in \mathcal{D}$. It ensues easily from Definition 6 that if $(1 >)\, \alpha \geq p_i$, $VaR_\alpha(X_i) = 0$, while if $0 < \alpha < p_i$, then $VaR_\alpha(X_i) = 1/r$.

To prove the 'if' implication, let $\alpha \geq \max\{p_1, \ldots, p_n\}$. It is sufficient by Definition 3 to show that, for some choice of the $X_i \in \mathcal{D}$ and of the corresponding s_i, the ensuing random number $\overline{G}$ is negative. In fact, select $X_1, \ldots, X_n$ and put $s_1 = \ldots = s_n = 1$. We obtain

$$\overline{G} = \sum_{i=1}^{n} s_i \left(\overline{P}\left(-X_i/r\right) + X_i/r\right) = \sum_{i=1}^{n} -|e_i|/r$$

$$= -1/r \sum_{i=1}^{n} |e_i| = -1/r < 0.$$

To prove the 'only if' part, let $p_h = \max\{p_1, \ldots, p_n\}$. Since $P(X_i \leq X_i(e_h)) = P(X_i \leq 0) = 1$ for $i \neq h$, it is $p_h = \min_{i=1,\ldots,n} P(X_i \leq X_i(e_h))$. Then, if $\alpha < p_h$, VaR_α avoids sure loss by Proposition 4 below. ■

In the example above VaR_α avoids sure loss if α is sufficiently small. The next proposition shows that this happens also in more general situations.

Proposition 4 *Let $\mathcal{D} = \{X_i\}_{i \in I}$ be a family of arbitrary (bounded) random numbers, $I\!P$ a partition whose atoms describe all values for X_i ($i \in I$) which are jointly possible and P a probability distribution on $I\!P$. If, for some $\omega \in I\!P$, condition*

$$0 < \alpha < \inf_{i \in I} P(X_i \leq X_i(\omega)) \tag{8}$$

holds, then VaR_α avoids sure loss on $\mathcal{D}$.

Proof Considering Definition 3, we prove that the upper prevision on $\mathcal{D}^* = \{-X/r : X \in \mathcal{D}\}$ such that $\overline{P}(-X/r) = VaR_\alpha(X)\ \forall X \in \mathcal{D}$ avoids sure loss. By Definition 1, we have to show that for each n, $X_1, \ldots, X_n \in \mathcal{D}$, $s_1 \ldots, s_n \geq 0$

$$\overline{G} = \sum_{i=1}^{n} s_i \left(VaR_\alpha\left(X_i\right) - \left(-X_i/r\right)\right) = \sum_{i=1}^{n} s_i \left(X_i/r - q_\alpha^+\left(X_i/r\right)\right) \tag{9}$$

has at least one non-negative value. In fact, since for $i = 1, \ldots, n$

$$P(X_i \leq X_i(\omega)) = P\left(X_i/r \leq X_i(\omega)/r\right) > \alpha$$

we get

$$q_\alpha^+(X_i/r) \leq X_i(\omega)/r.$$

If follows from the above inequality that $\overline{G}(\omega)$ is non-negative, being a sum of n non-negative terms. ■

Corollary 1 *If the cardinality of partition $I\!P$ in Proposition 4 is m, then VaR_α avoids sure loss for $\alpha < 1/m$.*

Proof Just apply Proposition 4 to an atom of $I\!P$ whose probability is not less than $1/m$ (there exists at least one such atom). ■

3.1.1 Comment

When we know the distribution functions of each X_i ($i \in I$), by Proposition 4 the upper bound for α can be raised to $\alpha < \sup_{\omega \in I\!P} \inf_{i \in I} P(X_i \leq X_i(\omega))$. When the partition $I\!P$ is finite, the sufficient condition of Corollary 1 does not require, on the contrary, assessing any probability evaluation on $I\!P$. It is not difficult to see, using Proposition 4, that VaR can avoid sure loss in many practical circumstances.

At any rate, the most interesting implication is that getting a more prudent and an at least avoiding sure loss risk evaluation are matching goals when VaR_α is the risk measure: one tends to obtain both of them by lowering α.

3.1.2 A note on the 'average risk'

Other well-known risk measures may always avoid sure loss although they are not coherent, therefore being in a sense closer then VaR to coherent risk measures. One such measure, mentioned in [2] and whose usage in life insurance is quite old, is the 'average risk' ('mittleres Risiko') ρ_{AR}, which could be defined in our framework on a set $\mathcal{D}$ of random numbers putting, for each $X \in \mathcal{D}$, $\rho_{AR}(X) = P(X^-)$, where $X^- = \max\{-X, 0\}$ and P is a precise coherent prevision or an expectation, if a joint distribution on all $X \in \mathcal{D}$ is given (as required in the original formulation of this measure). The extra assumption $r = 1$ is made. Since $X^- \geq -X$, then $P(X^-) \geq P(-X)$ and hence ρ_{AR} avoids sure loss by Theorem 1.

3.2 Coherent corrections of risk measures avoiding sure loss

It follows from Definition 3 that the problem of correcting risk measures avoiding sure loss into coherent ones can be viewed as the problem of correcting upper previsions that avoid sure loss.

As mentioned in 2), Section 2.2, an upper prevision $\overline{P}$ given on $\mathcal{D}$ that avoids sure loss can be corrected by building its natural extension $\overline{P}_E$. Formally, the natural extension can be defined for X in any set of random numbers $\mathcal{D}' \supset \mathcal{D}$ as

$$\overline{P}_E(X) = \inf \mathcal{N}$$

where, putting $\overline{g}_i = \overline{P}(X_i) - X_i$, it is

$$\mathcal{N} = \left\{\alpha : \alpha - X \geq \sum_{i=1}^{n} \lambda_i \overline{g}_i, \text{ for some } n \geq 0, X_i \in \mathcal{D}, \lambda_i \geq 0, \alpha \in I\!R\right\}.$$

As is well known [9], the natural extension satisfies the following properties:

a) $\overline{P}_E$ is a coherent upper prevision on $\mathcal{D}'$;

b) $\overline{P}_E(X) \leq \overline{P}(X)$, $\forall X \in \mathcal{D}$;

c) if $\overline{P}^*$ is a coherent upper prevision on $\mathcal{D}'$ such that $\overline{P}^*(X) \leq \overline{P}(X)$, $\forall X \in \mathcal{D}$, then $\overline{P}^*(X) \leq \overline{P}_E(X)$, $\forall X \in \mathcal{D}'$;

d) $\overline{P}$ is coherent if and only if $\overline{P}_E$ is the maximal coherent extension of $\overline{P}$ ($\overline{P}_E = \overline{P}$ on $\mathcal{D}$);

e) $\overline{P}$ avoids sure loss if and only if its natural extension is finite.

It follows from e) that it is essential for a risk measure ρ to avoid sure loss, if we wish to be able to replace it by its natural extension. This can be done correcting $\rho(X) = \overline{P}(-X/r)$ into $\rho^*(X) = \overline{P}_E(-X/r)$. From a), ρ^* is coherent and, from c), it is the *least-committal* coherent correction of $\overline{P}$.

Property b) tells us that the natural extension $\rho^*(X)$ of $\rho(X)$ is a less prudential risk estimate then $\rho(X)$.

In fact, it requires (the same or) a smaller amount than $\rho(X)$ to be added to a non-desirable risk X to make it desirable (cfr. the end of Section 2.3) and this might be a motivation for seeking other corrections of an incoherent $\rho(X)$; the matter is partly discussed below in this section. Anyway, being least-committal means that the natural extension of $\rho(X)$ is the more prudential coherent risk estimate among those which are less prudential than $\rho(X)$.

The problem discussed in this section has already been tackled in [2], Section 4.2, without resorting to the theory of imprecise previsions. In our framework, it can be proved that conditions 4.1 and 4.2 in [2] are equivalent, respectively, to Definition 3 (ASL) and Definition 4 (coherence); further, the quantity $\rho_\psi(X)$, Proposition 4.2, corresponds to ρ^* as defined above.

Operationally, when $\mathcal{D}$ is finite and all X in $\mathcal{D}$ are simple random numbers, the problem of finding the natural extension can be solved with linear programming techniques, similar to those discussed, for instance, in [6, 8, 11].

It is anyway important to observe that there might be some practical constraints which prevent from employing the natural extension correction of a risk measure, as noted in [2] referring to VaR. For instance, a regulator might impose a lower bound for any correction ρ^* of an incoherent VaR, like $\rho^* \geq VaR_\alpha$. This appears to be a peculiar feature of the application of imprecise previsions theory to risk problems; of course, if this is the case the correction problem has generally no unique solution. Consider, for instance, a partition $I\!P = \{e_1, e_2, e_3\}$ such that $P(e_1) = 0.02$, $P(e_2) = 0.04$ and let $X_1 = -2|e_1|$, $X_2 = -|e_2|$, $\mathcal{D} = \{X_1, X_2, X_1 + X_2\}$. If $\alpha = 5\%$ and $r = 1$, it is $VaR_\alpha(X_1) = VaR_\alpha(X_2) = 0$, whilst $VaR_\alpha(X_1 + X_2) = 1$, so that VaR_α is not coherent on $\mathcal{D}$ (although it avoids sure loss by Corollary 1) for lack of subadditivity. Therefore, an investor willing to assign a ρ^* as small as possible, without violating the bounds imposed by the regulator, should choose a risk measure ρ^* on $\mathcal{D}$ satisfying $\rho^*(X_1) \geq 0$, $\rho^*(X_2) \geq 0$, $\rho^*(X_1 + X_2) = \rho^*(X_1) + \rho^*(X_2) = 1$ (it can be easily seen that such a ρ^* is coherent). In this simple example, $\rho^*(X_1)$ can be given any value in $[0, 1]$. The choice of a specific $\rho^*(X_1)$ depends on additional considerations. For instance, the loss X_1 may cause is twice as much as the loss due to X_2, which suggests overweighting $\rho^*(X_1)$, but event $X_2 = -1$ is twice as likely as $X_1 = -2$, which suggests underweighting $\rho^*(X_1)$. The ultimate choice for $\rho^*(X_1)$ is not so evident.

4 Conclusions

In this paper we have outlined an approach to risk measures based on the theory of coherent imprecise previsions. Such an approach leads naturally to a more general definition of coherent risk measure and provides useful tools to investigate many properties of these as well as other risk measures. In particular, we discussed whether VaR satisfies a weaker consistency condition named ASL, giving sufficient conditions for this.

In our opinion, there is scope for further extending the use of imprecise previsions in this area. In particular, the theory of imprecise previsions has been studied also in the case of conditional previsions. From this and the results in Section 2 it appears that a related notion of coherent conditional risk measure could be introduced as a next generalization.

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