Probabilistic Logic under Coherence: Complexity and Algorithms

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Abstract

We study probabilistic logic under the viewpoint of the coherence principle of de Finetti. In detail, we explore the relationship between coherence-based and classical model-theoretic probabilistic logic. Interestingly, we show that the notions of g-coherence and of g-coherent entailment can be expressed by combining notions in model-theoretic probabilistic logic with concepts from default reasoning. Using these results, we analyze the computational complexity of probabilistic reasoning under coherence. Moreover, we present new algorithms for deciding g-coherence and for computing tight g-coherent intervals, which reduce these tasks to standard reasoning tasks in model-theoretic probabilistic logic. Thus, efficient techniques for model-theoretic probabilistic reasoning can immediately be applied for probabilistic reasoning under coherence, for example, column generation techniques. We then describe two other interesting techniques for efficient model-theoretic probabilistic reasoning in the conjunctive case.

Keywords. Conditional probability assessments, probabilistic logic, g-coherence, g-coherent entailment, complexity and algorithms.

1 Introduction

The probabilistic treatment of uncertainty plays an important role in many applications of knowledge representation and reasoning. Often, we need to reason with uncertain information under partial knowledge and then the use of precise probabilistic assessments seems unrealistic. Moreover, the family of uncertain quantities at hand has often no particular algebraic structure.

In such cases, a general approach is obtained by using (conditional and/or unconditional) probabilistic constraints, based on the coherence principle of de Finetti and suitable generalizations of it (Biazzo and Gilio [2], Coletti [10], Coletti and Scozzafava [11, 14, 13], Gilio [22, 23], Gilio and Scozzafava [24], and Scozzafava [34]), or on similar principles which have been adopted for lower and upper probabilities (Pelessoni and Vicig [33], Vicig [35], and Walley [36]). Two important aspects in dealing with uncertainty are: (i) checking the consistency of a probabilistic assessment; and (ii) the propagation of a given assessment to further uncertain quantities. The problem of reducing or eliminating the computational difficulties of (i) and (ii) has been recently investigated by Biazzo *et al.* [5, 6], Capotorti and Vantaggi [9], Capotorti *et al.* [8], and Coletti and Scozzafava [12].

Another approach to handle constraints for probabilities is model-theoretic probabilistic logic, whose roots go back to Boole's book of 1854 "The Laws of Thought" [7]. There is a wide spectrum of formal languages that have been explored in probabilistic logic, which ranges from constraints for unconditional and conditional events (see especially the work by Nilsson [32], Dubois *et al.* [15], Amarger *et al.* [1], and Frisch and Haddawy [18]) to rich languages that specify linear inequalities over events (Fagin *et al.* [17]). The reasoning methods in probabilistic logic can be roughly divided into local approaches based on local inference rules and global ones using linear optimization techniques (see especially [31, 30] on the issue of local versus global approaches). As shown by Georgakopoulos *et al.* [21], deciding satisfiability and logical consequence in probabilistic logic is NP- and co-NP-complete, and thus intractable. Moreover, as recently shown in [28], deciding and computing tight logical consequences is complete for the classes D^{P} and $\mathrm{F}\Delta_2^{\mathrm{P}}$, respectively. Substantial research efforts were directed towards efficient techniques for reasoning in probabilistic logic. In particular, column generation techniques from the area of linear optimization have been successfully used to solve large problem instances (see the work by Jaumard *et al.* [27] and Hansen *et al.* [26]). Other techniques, which may be described as problem transformations on the language level, have been successfully applied in probabilistic logic programming [28]. Moreover, a global approach for the conjunctive case, which characterizes a reduced set of variables, has been presented in [29].

We point out that in model-theoretic probabilistic logic, for every conditional constraint $\alpha \leq P(\psi|\phi) \leq \beta$ in the given probabilistic knowledge base, the conditional probability $P(\psi|\phi)$ is looked at as the ratio of $P(\psi \wedge \phi)$ and $P(\phi)$, so that it is defined only if $P(\phi) > 0$. On the contrary, as well known, within the approach of de Finetti: (i) one can directly assess conditional probabilities, with no need of defining them as ratios; (ii) no theoretical problem arises when the probabilities of some (or possibly all) *conditioning* events are (judged equal to) zero; and (iii) by exploiting the zero probabilities, the computational difficulties may be reduced (or even eliminated).

Coherence-based and model-theoretic probabilistic reasoning have been explored quite independently from each other by two different research communities. For this reason, the relationship between the two areas has not been studied in depth so far. Our work in [3] and the current paper aim at filling this gap. More precisely, our research is guided by the following two questions:

- Which is the semantic relationship between coherence-based and model-theoretic probabilistic reasoning?
- Can algorithms that have been developed for efficient reasoning in one area also be used in the other area?

Interestingly, it turns out that the answers to these two questions are closely related to the area of default reasoning from conditional knowledge bases [19]. Roughly speaking, coherence-based probabilistic reasoning can be understood as a combination of model-theoretic probabilistic logic with concepts from default reasoning.

That is, deciding coherence and computing tight intervals under coherence can be reduced to standard reasoning tasks in model-theoretic probabilistic logic. Thus, efficient techniques for model-theoretic probabilistic reasoning can be applied for probabilistic reasoning under coherence.

It is important to point out that the existence of such reductions to model-theoretic probabilistic reasoning does not imply that we actually do not need probabilistic reasoning under coherence. Model-theoretic probabilistic reasoning and probabilistic reasoning under coherence are two distinct concepts, and they both clearly have their own area of application. Our reductions now reveal the semantic relationships between the two formalisms, and thus help to delineate these areas of applications.

Our work in [3] explores the semantic aspects of this relationship. In this paper, we focus on its computational implications for coherence-based probabilistic reasoning.

The main contributions can be summarized as follows.

- We define a coherence-based probabilistic logic. In detail, we define a formal language of logical and conditional constraints, which are defined on arbitrary families of conditional events. We then define the notions of generalized coherence (*g-coherence*), g-coherent consequence, and tight g-coherent consequence for this language.
- We recall from [3] the relationship between g-coherence and g-coherent entailment, on the one hand, and satisfiability and logical entailment, on the other hand.
- We analyze the computational complexity of deciding g-coherence and g-coherent consequence. To our knowledge, no such results have been derived so far.
- We present new algorithms for deciding g-coherence and for computing tight g-coherent consequences. Based on concepts from default reasoning, they reduce checking g-coherence and computing tight g-coherent intervals to standard tasks in model-theoretic probabilistic logic, which can be reduced to linear optimization problems.
- As a consequence of the previous result, all the techniques that have been developed for efficient model-theoretic probabilistic reasoning can now also be applied in probabilistic reasoning under g-coherence, for example, column generation techniques [21, 27, 26].
- We describe two other techniques for the *conjunctive case*, namely (i) removing inactive logical and conditional constraints, and (ii) producing a reduced set of variables for the linear optimization problems. The former is inspired by [28, 16], while the latter is taken from [29].
- Interestingly, it turns out that the technique of generating a reduced set of variables taken from [29] can be characterized using the notion of random gain.

The rest of this paper is organized as follows. Section 2 introduces the formal background of this work, and recalls the semantic relationship between coherence-based and model-theoretic probabilistic reasoning from [3]. In Sections 3 and 4, we present our complexity results and new algorithms for coherence-based probabilistic reasoning. Section 5 describes two techniques for an increased efficiency of reasoning in model-theoretic probabilistic logic in the conjunctive case. In Section 6, we summarize the main results and give an outlook on future research.

Note that detailed proofs of all results are given in the extended paper [4].

2 Formal Background

In this section, we first introduce some technical preliminaries. We then briefly describe precise and imprecise probability assessments under coherence, and a model-theoretic probabilistic logic of conditional constraints. We finally define our coherence-based probabilistic logic.

2.1 Preliminaries

We assume a nonempty set of *basic events* Φ. We use $\perp$ and $\top$ to denote *false* and *true*, respectively. The set of *events* is the closure of $\Phi \cup \{\perp, \top\}$ under the Boolean operations $\neg$ and $\wedge$. That is, each element of $\Phi \cup \{\perp, \top\}$ is an event, and if ϕ and ψ are events, then also $(\phi \wedge \psi)$

and $\neg\phi$. We use $(\phi \vee \psi)$ and $(\psi \Leftarrow \phi)$ to abbreviate $\neg(\neg\phi \wedge \neg\psi)$ and $\neg(\phi \wedge \neg\psi)$, respectively, and adopt the usual conventions to eliminate parentheses. We often denote by $\bar{\phi}$ the negation $\neg\phi$, and by $\phi\psi$ the conjunction $\phi \wedge \phi$. A *logical constraint* is an event of the form $\psi \Leftarrow \phi$.

A *world* I is a truth assignment to the basic events in Φ (that is, a mapping $I\colon\ \Phi \to \{\mathbf{false}, \mathbf{true}\}$), which is extended to all events as usual (that is, $(\phi \wedge \psi)$ is true in I iff ϕ and ψ are true in I, and $\neg\phi$ is true in I iff ϕ is not true in I). We use $\mathcal{I}_\Phi$ to denote the set of all worlds for Φ. We often identify the truth values **false** and $\mathbf{true}$ with the real numbers 0 and 1, respectively. A world I *satisfies* an event ϕ, or I is a *model* of ϕ, denoted $I \models \phi$, iff $I(\phi) = \mathbf{true}$. I *satisfies* a set of events L, or I is a *model* of L, denoted $I \models L$, iff I is a model of all $\phi \in L$. An event ϕ (resp., a set of events L) is *satisfiable* iff a model of ϕ (resp., L) exists. An event ψ is a *logical consequence* of ϕ (resp., L), denoted $\phi \models \psi$ (resp., $L \models \psi$), iff each model of ϕ (resp., L) is also a model of ψ. We use $\phi \not\models \psi$ (resp., $L \not\models \psi$) to denote that $\phi \models \psi$ (resp., $L \models \psi$) does not hold.

2.2 Probability Assessments

A *conditional event* is an expression of the form $\psi|\phi$ with events ψ and $\phi \neq \bot$. It can be looked at as a three-valued logical entity, with values **true**, or **false**, or **indeterminate**, according to whether ψ and ϕ are true, or ψ is false and ϕ is true, or ϕ is false, respectively. Notice that $\psi|\phi$ coincides with $\psi \wedge \phi|\phi$. More generally, $\psi_1|\phi_1$ and $\psi_2|\phi_2$ coincide iff $\psi_1 \wedge \phi_1 = \psi_2 \wedge \phi_2$ and $\phi_1 = \phi_2$.

A *probability assessment* (L, A) on a set of conditional events $\mathcal{E}$ consists of a set of logical constraints L, and a mapping A that assigns each $\varepsilon \in \mathcal{E}$ a real number in $[0, 1]$. Informally, L describes logical relationships, while A represents probabilistic knowledge.

For $\{\psi_1|\phi_1, \ldots, \psi_n|\phi_n\} \subseteq \mathcal{E}$ with $n \geq 1$ and n real numbers $s_1, \ldots, s_n$, let the mapping $G\colon \mathcal{I}_\Phi \to \mathbf{R}$ be defined as follows. For every $I \in \mathcal{I}_\Phi$:

$$G(I) \;=\; \sum_{i=1}^{n} s_i \cdot I(\phi_i) \cdot (I(\psi_i) - A(\psi_i|\phi_i))\,.$$

Intuitively, G can be interpreted as the random gain corresponding to a combination of n bets of amounts $s_1 \cdot A(\psi_1|\phi_1), \ldots, s_n \cdot A(\psi_n|\phi_n)$ on $\psi_1|\phi_1, \ldots, \psi_n|\phi_n$ with stakes $s_1, \ldots, s_n$. In detail, to bet on $\psi_i|\phi_i$, one pays an amount of $s_i \cdot A(\psi_i|\phi_i)$, and one gets back the amount of s_i, 0, and $s_i \cdot A(\psi_i|\phi_i)$, when $\psi_i \wedge \phi_i$, $\neg\psi_i \wedge \phi_i$, and $\neg\phi_i$, respectively, turns out to be true.

The following notion of *coherence* now assures that it is impossible (for both the gambler and the bookmaker) to have *uniform loss* under this betting scheme. A probability assessment (L, A) on a set of conditional events $\mathcal{E}$ is *coherent* iff for every $\{\psi_1|\phi_1, \ldots, \psi_n|\phi_n\} \subseteq \mathcal{E}$ with $n \geq 1$ and for all real numbers $s_1, \ldots, s_n$, the following holds:

$$\max_{\substack{I \in \mathcal{I}_\Phi,\, I \models L \\ I \models \phi_1 \vee \cdots \vee \phi_n}} \;\sum_{i=1}^{n} s_i \cdot I(\phi_i) \cdot (I(\psi_i) - A(\psi_i|\phi_i)) \;\geq\; 0\,.$$

2.3 Imprecise Probability Assessments

An *imprecise probability assessment* (L, A) on a set of conditional events $\mathcal{E}$ consists of a set of logical constraints L and a mapping A that assigns each $\varepsilon \in \mathcal{E}$ an interval $[l, u] \subseteq [0, 1]$ with $l \leq u$. We say (L, A) is *g-coherent* iff there exists a coherent precise probability assessment $(L, A^\star)$ on $\mathcal{E}$ such that $A^\star(\varepsilon) \in A(\varepsilon)$ for all $\varepsilon \in \mathcal{E}$.

Given a set of logical constraints L, and a set of conditional events $\mathcal{E} = \{\varepsilon_1, \ldots, \varepsilon_n\}$, denote by $R_L(\mathcal{E})$ the set of all mappings r that assign each $\varepsilon_i = \psi_i|\phi_i \in \mathcal{E}$ a member of $\{\psi_i \wedge \phi_i, \neg\psi_i \wedge \phi_i, \neg\phi_i\}$ such that (i) $L \cup \{r(\varepsilon_i) \mid \varepsilon_i \in \mathcal{E}\}$ is satisfiable, and (ii) $r(\varepsilon_i) \neq \neg\phi_i$ for some $i \in \{1, \ldots, n\}$. For such mappings r and events ϕ, we use $r \models \phi$ to abbreviate $r(\varepsilon_1) \wedge \cdots \wedge r(\varepsilon_n) \models \phi$.

Theorem 2.1 (Gilio [22]) *An imprecise probability assessment (L, A) on a set of conditional events $\mathcal{E}$ is g-coherent iff for every $\mathcal{E}_n = \{\psi_1|\phi_1, \ldots, \psi_n|\phi_n\} \subseteq \mathcal{E}$ with $n \geq 1$, the following system of linear constraints over the variables λ_r $(r \in R)$, where $R = R_L(\mathcal{E}_n)$, is solvable:*

$$\begin{aligned} \sum_{r \in R} \lambda_r \cdot p_{ri} \;&\geq\; l_i \quad (\textit{for all } i \in \{1, \ldots, n\}) \\ \sum_{r \in R} \lambda_r \cdot q_{ri} \;&\leq\; u_i \quad (\textit{for all } i \in \{1, \ldots, n\}) \\ \sum_{r \in R} \lambda_r \;&=\; 1 \\ \lambda_r \;&\geq\; 0 \quad (\textit{for all } r \in R)\,, \end{aligned} \tag{1}$$

where l_i and u_i are defined by $A(\psi_i|\phi_i) = [l_i, u_i]$ for all $i \in \{1, \ldots, n\}$, and p_{ri} and q_{ri} are defined as follows for all $r \in R$ and $i \in \{1, \ldots, n\}$:

$$p_{ri}\ \ (\textit{resp., } q_{ri}) \;=\; \begin{cases} 1, & \textit{if } r \models \psi_i \wedge \phi_i \\ 0, & \textit{if } r \models \neg\psi_i \wedge \phi_i \\ l_i\ \ (\textit{resp., } u_i), & \textit{if } r \models \neg\phi_i\,. \end{cases} \tag{2}$$

Equivalent results have been obtained by Coletti [10].

Let (L, A) be a g-coherent imprecise probability assessment on a set of conditional events $\mathcal{E}$. The imprecise probability assessment $[l, u]$ on a conditional event γ is called a *g-coherent consequence* of (L, A) iff $A^\star(\gamma) \in [l, u]$ for every g-coherent precise probability assessment $A^\star$ on $\mathcal{E} \cup \{\gamma\}$ such that $A^\star(\varepsilon) \in A(\varepsilon)$ for all $\varepsilon \in \mathcal{E}$. It is a *tight g-coherent consequence* of (L, A) iff l (resp., u) is the infimum (resp., supremum) of $A^\star(\gamma)$ subject to all g-coherent precise probability assessments $A^\star$ on $\mathcal{E} \cup \{\gamma\}$ such that $A^\star(\varepsilon) \in A(\varepsilon)$ for all $\varepsilon \in \mathcal{E}$.

2.4 Probabilistic Logic

In the rest of this paper, we assume that Φ is finite. A *conditional constraint* is an expression $(\psi|\phi)[l,u]$ with real numbers $l, u \in [0,1]$ and events ψ and ϕ. We call ϕ its *antecedent* and ψ its *consequent*. A probabilistic knowledge base $KB = (L, P)$ consists of a finite set of logical constraints L, and a finite set of conditional constraints P such that (i) $l \leq u$ for all $(\psi|\phi)[l,u] \in P$, and (ii) $\psi_1|\phi_1 \neq \psi_2|\phi_2$ for all distinct $(\psi_1|\phi_1)[l_1,u_1], (\psi_2|\phi_2)[l_2,u_2] \in P$.

A *probabilistic interpretation* Pr is a probability function on $\mathcal{I}_\Phi$ (that is, a mapping $Pr\colon \mathcal{I}_\Phi \rightarrow [0,1]$ such that all $Pr(I)$ with $I \in \mathcal{I}_\Phi$ sum up to 1). The *probability* of an event ϕ in the probabilistic interpretation Pr, denoted $Pr(\phi)$, is defined as follows:

$$Pr(\phi) \;=\; \sum_{I \in \mathcal{I}_\Phi,\, I \models \phi} Pr(I)\,.$$

For events ϕ and ψ with $Pr(\phi) > 0$, we use $Pr(\psi|\phi)$ to abbreviate $Pr(\psi \wedge \phi)\,/\,Pr(\phi)$. The *truth* of logical and conditional constraints F in a probabilistic interpretation Pr, denoted $Pr \models F$, is defined as follows:

- $Pr \models \psi \Leftarrow \phi$ iff $Pr(\psi \wedge \phi) = Pr(\phi)$.
- $Pr \models (\psi|\phi)[l,u]$ iff $Pr(\phi) = 0$ or $Pr(\psi|\phi) \in [l,u]$.

We say Pr *satisfies* a logical or conditional constraint F, or Pr is a *model* of F, iff $Pr \models F$. We say Pr *satisfies* a set of logical and conditional constraints $\mathcal{F}$, or Pr is a *model* of $\mathcal{F}$, denoted $Pr \models \mathcal{F}$, iff Pr is a model of all $F \in \mathcal{F}$. We say that $\mathcal{F}$ is *satisfiable* iff a model of $\mathcal{F}$ exists.

We next define the notion of logical entailment. A conditional constraint $F = (\psi|\phi)[l,u]$ is a *logical consequence* of a set of logical and conditional constraints $\mathcal{F}$, denoted $\mathcal{F} \models F$, iff each model of $\mathcal{F}$ is also a model of F. It is a *tight logical consequence* of $\mathcal{F}$, denoted $\mathcal{F} \models_{\mathit{tight}} F$, iff l (resp., u) is the infimum (resp., supremum) of $Pr(\psi|\phi)$ subject to all models Pr of $\mathcal{F}$ with $Pr(\phi) > 0$. Note that we define $l = 1$ and $u = 0$, when $\mathcal{F} \models (\phi|\top)[0,0]$.

A probabilistic knowledge bases $KB = (L, P)$ is *satisfiable* iff $L \cup P$ is satisfiable. A conditional constraint $(\psi|\phi)[l,u]$ is a *logical consequence* of KB, denoted $KB \models (\psi|\phi)[l,u]$, iff $L \cup P \models (\psi|\phi)[l,u]$. It is a *tight logical consequence* of KB, denoted $KB \models_{\mathit{tight}} (\psi|\phi)[l,u]$, iff $L \cup P \models_{\mathit{tight}} (\psi|\phi)[l,u]$.

2.5 Probabilistic Logic under Coherence

Every imprecise probability assessment $IP = (L, A)$ with finite L on a finite set of conditional events $\mathcal{E}$ can be represented by the following probabilistic knowledge base:

$$KB_{IP} = (L, \{(\psi|\phi)[l,u] \mid \psi|\phi \in \mathcal{E},\ A(\psi|\phi) = [l,u]\})\,.$$

Conversely, every probabilistic knowledge base $KB = (L, P)$ can expressed by the following imprecise probability assessment $IP_{KB} = (L, A_{KB})$ on $\mathcal{E}_{KB}$:

$$\begin{aligned} A_{KB} &= \{(\psi|\phi, [l,u]) \mid (\psi|\phi)[l,u] \in KB\}\,, \\ \mathcal{E}_{KB} &= \{\psi|\phi \mid \exists\, l,u \in [0,1]\colon (\psi|\phi)[l,u] \in KB\}\,. \end{aligned}$$

A probabilistic knowledge base KB is said *g-coherent* iff IP_{KB} is g-coherent. Given a g-coherent probabilistic knowledge base KB and a conditional constraint $(\psi|\phi)[l,u]$, we say $(\psi|\phi)[l,u]$ is a *g-coherent consequence* of KB, denoted $KB \mid\!\sim (\psi|\phi)[l,u]$, iff $\{(\psi|\phi, [l,u])\}$ is a g-coherent consequence of IP_{KB}. It is a *tight g-coherent consequence* of KB, denoted $KB \mid\!\sim_{\mathit{tight}} (\psi|\phi)[l,u]$, iff $\{(\psi|\phi, [l,u])\}$ is a tight g-coherent consequence of IP_{KB}.

The following is an immediate implication.

Theorem 2.2 *Let $KB = (L, P)$ be a g-coherent probabilistic knowledge base, and let $(\psi|\phi)[l,u]$ be a conditional constraint. Then, $KB \mid\!\sim (\psi|\phi)[l,u]$ iff $(L, P \cup \{(\psi|\phi)[p,p]\})$ is not g-coherent for all $p \in [0,l) \cup (u,1]$.*

2.6 Relationship to Probabilistic Logic

We now recall some of our results obtained in [3], which concern characterizations of the notions of g-coherence and of g-coherent entailment in terms of the notions of satisfiability and of logical entailment.

We adopt the following terminology from the area of default reasoning [19]. A probabilistic interpretation Pr *verifies* a conditional constraint $(\psi|\phi)[l,u]$ iff $Pr(\phi) > 0$ and $Pr \models (\psi|\phi)[l,u]$. A set of conditional constraints P *tolerates* a conditional constraint F under a set of logical constraints L iff there exists a model of $L \cup P$ that verifies F.

The following theorem shows that g-coherence has a characterization similar to ε-consistency in default reasoning [25]. This result follows from Theorem 2.1.

Theorem 2.3 *Let $KB = (L, P)$ be a probabilistic knowledge base. Then, KB is g-coherent iff there is an ordered partition $(P_0, \ldots, P_k)$ of P such that each P_i is the set of all elements in $\bigcup_{j=i}^{k} P_j$ tolerated under L by $\bigcup_{j=i}^{k} P_j$.*

We next express the notion of g-coherent entailment in terms of the notion of logical entailment.

For a probabilistic knowledge base $KB = (L, P)$ and an event α, we use $P_\alpha(KB)$ to denote the set of all subsets $P_n = \{(\psi_1|\phi_1)[l_1,u_1], \ldots, (\psi_n|\phi_n)[l_n,u_n]\}$ of P such that every model Pr of $L \cup P_n$ with $Pr(\phi_1 \vee \cdots \vee \phi_n \vee \alpha) > 0$ satisfies $Pr(\alpha) > 0$.

The following lemma shows that $P_\alpha(KB)$ has a unique greatest element with respect to set inclusion.

Lemma 2.4 *Let $KB = (L, P)$ be a g-coherent probabilistic knowledge base, and let α be an event. Then, $P_\alpha(KB)$ contains a unique greatest element.*

The next theorem shows the crucial result that *g-coherent entailment from KB can be reduced to logical entailment from the greatest element in $P_\alpha(KB)$*.

Theorem 2.5 *Let $KB = (L, P)$ be a g-coherent probabilistic knowledge base, and let $F = (\beta|\alpha)[l, u]$ be a conditional constraint. Let $KB^\star = (L, P^\star)$, where $P^\star$ is the greatest element in $P_\alpha(KB)$. Then,*

(a) $KB \mathrel{|\!\sim} F$ *iff* $KB^\star \models F$.

(b) $KB \mathrel{|\!\sim}_{tight} F$ *iff* $KB^\star \models_{tight} F$.

Thus, computing tight g-coherent consequences can be reduced to computing tight logical consequences from the greatest element $P^\star$ in $P_\alpha(KB)$. The following theorem shows how $P^\star$ can be characterized and thus computed.

Theorem 2.6 *Let $KB = (L, P)$ be a g-coherent probabilistic knowledge base and α be an event. Let $P^\star \subseteq P$ and $(P_0, \ldots, P_k)$ be an ordered partition of $P \backslash P^\star$ such that (i) every P_i is the set of all elements in $P_i \cup \cdots \cup P_k \cup P^\star$ tolerated under $L \cup \{\bot \Leftarrow \alpha\}$ by $P_i \cup \cdots \cup P_k \cup P^\star$, and (ii) no member of $P^\star$ is tolerated under $L \cup \{\bot \Leftarrow \alpha\}$ by $P^\star$. Then, $P^\star$ is the greatest element in $P_\alpha(KB)$.*

3 Complexity

In this section, we analyze the computational complexity of deciding g-coherence and g-coherent consequence.

It turns out that deciding g-coherence and g-coherent consequence are NP- and co-NP-complete, respectively, and thus intractable, even in very restricted cases. Hence, we cannot hope for algorithms that efficiently solve every problem instance. But we can still hope for efficient special-case, average-case, and approximation algorithms.

This knowledge about the precise complexity of probabilistic reasoning under coherence is very useful for developing efficient algorithms. In particular, the above results show that g-coherence and g-coherent consequence can be polynomially translated into SAT instances, and thus be solved with existing sophisticated SAT packages.

In the sequel, let $KB = (L, P)$ be a probabilistic knowledge base. We assume that all bounds in P are rational numbers. A conditional constraint $(\psi|\phi)[l, u]$ is *atomic* iff ψ is a basic event, and ϕ is either $\top$ or a basic event. It is *1-conjunctive* iff ψ is a basic event, and ϕ is either $\top$ or a conjunction of basic events. A set of conditional constraints $\mathcal{F}$ is *atomic* (resp., *1-conjunctive*) iff all members of $\mathcal{F}$ are atomic (resp., 1-conjunctive).

The following theorem shows that the problem of deciding g-coherence is NP-complete. Membership in NP follows from a characterization of g-coherence similar to Theorem 2.3 and a small-model theorem in probabilistic logic. Hardness for NP is obtained by a polynomial reduction from the NP-complete graph 3-colorability problem.

Theorem 3.1 *Given $KB = (L, P)$, deciding whether KB is g-coherent is* NP-*complete. Hardness holds even if L is empty and P is atomic.*

The next theorem shows that deciding g-coherent consequence is co-NP-complete. Here, co-NP-membership follows from Theorems 2.2, 2.5, and 3.1 and a small-model theorem in probabilistic logic. Hardness for co-NP can be proved by a polynomial reduction from the complement of the NP-complete graph 3-colorability problem.

Theorem 3.2 *Given a g-coherent $KB = (L, P)$ and a conditional constraint F, deciding whether $KB \mathrel{|\!\sim} F$ is* co-NP-*complete. Hardness holds even if L is empty, P is 1-conjunctive, and $F = (b|\top)[l, u]$ with $b \in \Phi$.*

4 Algorithms

Our results on the relationship between coherence- and model-based probabilistic logic in Section 2.6 open a new perspective on algorithms for deciding g-coherence and for computing tight g-coherent consequences. They show how these problems can be reduced to standard reasoning tasks in model-theoretic probabilistic logic.

That is, we get new algorithms for deciding g-coherence and computing tight g-coherent consequences, which are based on subprocedures for standard reasoning tasks in model-theoretic probabilistic logic. In the rest of this section, we first describe the new algorithms, and then discuss the tasks in model-theoretic probabilistic reasoning.

As for g-coherence, a previous algorithm by Gilio [22] is reformulated using terminology from the area of default reasoning. Interestingly, the resulting algorithm for deciding g-coherence is closely related to an algorithm for checking ε-consistency in default reasoning [25].[1]

As for tight g-coherent consequence, we have the important result that a tight interval under g-coherent entailment can be computed by first checking g-coherence, and then computing a tight interval under logical entailment.

The key features of these new algorithms can be briefly described as follows. First, they are surprisingly simple compared to existing algorithms. This helps to more deeply understand the semantic properties of g-coherence and g-coherent entailment (see [3] for more details). Second, they show that efficient techniques for model-theoretic probabilistic reasoning can be applied for probabilistic reasoning under coherence. We will illustrate this aspect in Section 5 by describing two such recent techniques.

[1] Note that a relationship between the algorithms in [22] and [25] was suggested first by Didier Dubois (personal communication).

Algorithm g-coherence

Input: Probabilistic knowledge base $KB = (L, P)$.
Output: "Yes", if KB is g-coherent, otherwise "No".

1. $Q := P$;
2. **repeat**
3. $\quad \Delta := \{F \in Q \mid F$ is tolerated under L by $Q\}$;
4. $\quad Q := Q \setminus \Delta$
5. **until** $\Delta = \emptyset$;
6. **if** $Q = \emptyset$ **then return** "Yes"
7. $\quad$ **else return** "No".

Figure 1: Algorithm g-coherence.

Algorithm tight-g-coherent-consequence

Input: Probabilistic knowledge base $KB = (L, P)$ and two events α and β.
Output: Unique pair of real numbers $(l, u) \in [0, 1] \times [0, 1]$ such that $KB \mid\!\sim_{tight} (\beta|\alpha)[l, u]$.

1. $Q := P$;
2. **repeat**
3. $\quad \Delta := \{F \in Q \mid F$ is tolerated under $L \cup \{\bot \Leftarrow \alpha\}$ by $Q\}$;
4. $\quad Q := Q \setminus \Delta$
5. **until** $\Delta = \emptyset$;
6. compute $l, u \in [0, 1]$ such that $L \cup Q \models_{tight} (\beta|\alpha)[l, u]$;
7. **return** (l, u).

Figure 2: Algorithm tight-g-coherent-consequence.

4.1 New Algorithms

In the sequel, let $KB = (L, P)$ be a probabilistic knowledge base, and let α and β be two events.

By Theorem 2.3, deciding whether KB is g-coherent can be done by Algorithm g-coherence (see Fig. 1): We first compute the set Δ_0 of all elements in $P_0 = P$ tolerated by P_0 under L. We then compute the set Δ_1 of all elements in $P_1 = P_0 \setminus \Delta_0$ tolerated by P_1 under L. We iterate this procedure until no element in $P_k = P_{k-1} \setminus \Delta_{k-1}$ is tolerated by P_k under L. Then, KB is g-coherent iff P_k is empty.

By Theorems 2.5 and 2.6, the real numbers $l, u \in [0, 1]$ such that $KB \mid\!\sim_{tight} (\beta|\alpha)[l, u]$ can be computed in a similar way by Algorithm tight-g-coherent-consequence (see Fig. 2): We first compute the set Δ_0 of all elements in $P_0 = P$ tolerated by P_0 under $L \cup \{\bot \Leftarrow \alpha\}$. We then compute the set Δ_1 of all elements in $P_1 = P_0 \setminus \Delta_0$ tolerated by P_1 under $L \cup \{\bot \Leftarrow \alpha\}$. We iterate this procedure until no element in $P_k = P_{k-1} \setminus \Delta_{k-1}$ is tolerated by P_k under $L \cup \{\bot \Leftarrow \alpha\}$. Then, we compute the real numbers $l, u \in [0, 1]$ such that $L \cup P_k \models_{tight} (\beta|\alpha)[l, u]$. They are the requested $l, u \in [0, 1]$ such that $KB \mid\!\sim_{tight} (\beta|\alpha)[l, u]$.

We now illustrate these algorithms by some examples.

Example 4.1 Let the probabilistic knowledge base $KB = (L, P)$ over $\Phi = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\}$ be given by:

$$(L, P) = (\emptyset, \{(\mathsf{B}|\mathsf{A})[0, .2], (\mathsf{C}|\mathsf{AB})[0, .2], (\mathsf{D}|\mathsf{C})[0, .2]\})\,.$$

Assume that we ask whether KB is g-coherent. Using Algorithm g-coherence, we then compute the set of all members of P tolerated by P under L. As this set already coincides with P, it follows that KB is g-coherent.

Assume now we ask for the real numbers $l, u \in [0, 1]$ such that $KB \mid\!\sim_{tight} (\mathsf{BCD}|\mathsf{A})[l, u]$. Using Algorithm tight-g-coherent-consequence, we then first compute the set of all members of P tolerated by P under $L \cup \{\bot \Leftarrow \mathsf{A}\}$. That is, we ask the following questions:

(1.1) Has $P \cup \{\bot \Leftarrow \mathsf{A}\}$ a model Pr with $Pr(\mathsf{A}) > 0$?
(1.2) Has $P \cup \{\bot \Leftarrow \mathsf{A}\}$ a model Pr with $Pr(\mathsf{AB}) > 0$?
(1.3) Has $P \cup \{\bot \Leftarrow \mathsf{A}\}$ a model Pr with $Pr(\mathsf{C}) > 0$?

The answers to (1.1) and (1.2) are (trivially) "No", while the answer to (1.3) is "Yes". Hence, we continue by computing the set of all members of $P' = P \setminus \{(\mathsf{D}|\mathsf{C})[0, .2]\}$ tolerated by P' under $L \cup \{\bot \Leftarrow \mathsf{A}\}$, by asking:

(2.1) Has $P' \cup \{\bot \Leftarrow \mathsf{A}\}$ a model Pr with $Pr(\mathsf{A}) > 0$?
(2.2) Has $P' \cup \{\bot \Leftarrow \mathsf{A}\}$ a model Pr with $Pr(\mathsf{AB}) > 0$?

As the answers to (2.1) and (2.2) are both (trivially) "No", the interval $[l, u] \subseteq [0, 1]$ with $KB \mid\!\sim_{tight} (\mathsf{BCD}|\mathsf{A})[l, u]$ is determined by $L \cup P' \models_{tight} (\mathsf{BCD}|\mathsf{A})[l, u]$. Numerically, this interval $[l, u]$ is given by $[0, 0.4]$. □

$$
\begin{aligned}
\sum_{r\in R,\, r\models\neg\psi\wedge\phi} -l\, y_r + \sum_{r\in R,\, r\models\psi\wedge\phi} (1-l)\, y_r \;&\geq\; 0 \quad (\text{for all } (\psi|\phi)[l,u]\in P,\; l>0)\\
\sum_{r\in R,\, r\models\neg\psi\wedge\phi} u\, y_r + \sum_{r\in R,\, r\models\psi\wedge\phi} (u-1)\, y_r \;&\geq\; 0 \quad (\text{for all } (\psi|\phi)[l,u]\in P,\; u<1)\\
\sum_{r\in R,\, r\models\alpha} y_r \;&=\; 1\\
y_r \;&\geq\; 0 \quad (\text{for all } r\in R)
\end{aligned}
\tag{3}
$$

Figure 3: System of linear constraints for Theorems 4.3 and 4.4.

The following is an example in which a tight g-coherent consequence coincides with a tight logical consequence.

Example 4.2 Let the probabilistic knowledge base $KB = (L, P)$ over $\Phi = \{\mathsf{A,B,C,D,E,F,G,H,I,J}\}$ be given by:

$$
\begin{aligned}
L \;=\; & \{\mathsf{G}\Leftarrow\mathsf{J}, \mathsf{H}\Leftarrow\mathsf{J}, \mathsf{I}\Leftarrow\mathsf{J}, \mathsf{A}\Leftarrow\mathsf{G}, \mathsf{B}\Leftarrow\mathsf{G}, \mathsf{C}\Leftarrow\mathsf{G},\\
& \mathsf{D}\Leftarrow\mathsf{I}, \mathsf{E}\Leftarrow\mathsf{I}, \mathsf{F}\Leftarrow\mathsf{I}\}\,,\\
P \;=\; & \{(\mathsf{J}|\mathsf{G})[.2,.3], (\mathsf{G}|\mathsf{A})[.2,.3], (\mathsf{G}|\mathsf{B})[.2,.3],\\
& (\mathsf{G}|\mathsf{C})[.2,.3], (\mathsf{J}|\mathsf{H})[.2,.3], (\mathsf{J}|\mathsf{I})[.2,.3],\\
& (\mathsf{I}|\mathsf{D})[.2,.3], (\mathsf{I}|\mathsf{E})[.2,.3], (\mathsf{I}|\mathsf{F})[.2,.3]\}\,.
\end{aligned}
$$

It is now easy to verify that every member of P is tolerated by P under L. Hence, KB is g-coherent.

Assume next we ask for the real numbers $l, u \in [0,1]$ such that $(\mathsf{J}|\mathsf{A})[l,u]$ is a g-coherent consequence of KB. Since no member of P is tolerated by P under $L \cup \{\bot \Leftarrow \mathsf{A}\}$, the interval $[l,u] \subseteq [0,1]$ such that $KB \mid\!\sim_{tight} (\mathsf{J}|\mathsf{A})[l,u]$ is determined by $L \cup P \models_{tight} (\mathsf{J}|\mathsf{A})[l,u]$. Numerically, this interval $[l,u]$ is given by $[0.04, 0.09]$. □

4.2 Tasks in Probabilistic Logic

The new algorithms in Section 4.1 reduce the problems of deciding g-coherence and of computing tight g-coherent consequences to the two standard tasks in model-theoretic probabilistic logic of deciding the existence of models that associate with a given event a positive probability (**PP**) and of computing tight logical consequences (**TLC**).

Task **PP** can be reduced to deciding whether a system of linear constraints is solvable. This is more formally expressed by the following well-known result.

Theorem 4.3 *Let $KB = (L,P)$ be a probabilistic knowledge base, and α be an event. Let $R = R_L(\mathcal{E}_{KB} \cup \{\alpha|\alpha\})$. Then, KB has a model Pr such that $Pr(\alpha) > 0$ iff the system of linear constraints* (3) *over the variables y_r ($r \in R$) is solvable (see Fig. 3).*

Task **TLC** is reducible to computing the optimal values of two linear programs, as the next well-known result shows.

Theorem 4.4 *Let $KB = (L,P)$ be a probabilistic knowledge base, and α, β be events. Let $R = R_L(\mathcal{E}_{KB} \cup \{\beta|\alpha\})$. Suppose that KB has a model Pr such that $Pr(\alpha) > 0$. Then, l (resp., u) such that $KB \models_{tight} (\beta|\alpha)[l,u]$ is given by the optimal value of the following linear program over the variables y_r ($r \in R$):*

$$
\textit{minimize (resp., maximize)} \sum_{r\in R,\, r\models\beta\wedge\alpha} y_r \quad \textit{subject to } (3)\,.
$$

We give an illustrative example.

Example 4.5 Consider again the probabilistic knowledge base $KB = (L,P)$ given in Example 4.1.

Assume we want to reply question (1.3). That is, has $P \cup \{\bot \Leftarrow \mathsf{A}\}$ a model Pr with $Pr(\mathsf{C}) > 0$? We now apply Theorem 4.3. The set $\mathcal{E} = \mathcal{E}_{KB} \cup \{\mathsf{C}|\mathsf{C}\}$ is given by:

$$
\mathcal{E} \;=\; \{\mathsf{B}|\mathsf{A}, \mathsf{C}|\mathsf{AB}, \mathsf{D}|\mathsf{C}, \mathsf{C}|\mathsf{C}\}\,.
$$

The set $R = R_{L\cup\{\bot\Leftarrow\mathsf{A}\}}(\mathcal{E})$ is represented as follows:

$$
R \;\hat{=}\; \{\bar{\mathsf{A}}\mathsf{C}\bar{\mathsf{D}}, \bar{\mathsf{A}}\mathsf{CD}\}\,.
$$

Thus, the system of linear constraints (3) is given by:

$$
\begin{aligned}
0.2\, y_{\bar{\mathsf{A}}\mathsf{C}\bar{\mathsf{D}}} - 0.8\, y_{\bar{\mathsf{A}}\mathsf{CD}} \;&\geq\; 0\\
y_{\bar{\mathsf{A}}\mathsf{C}\bar{\mathsf{D}}} + y_{\bar{\mathsf{A}}\mathsf{CD}} \;&=\; 1\\
y_{\bar{\mathsf{A}}\mathsf{C}\bar{\mathsf{D}}}, y_{\bar{\mathsf{A}}\mathsf{CD}} \;&\geq\; 0\,.
\end{aligned}
\tag{4}
$$

As (4) is solvable, the reply to question (1.3) is "Yes".

Assume next we want to compute the interval $[l,u] \subseteq [0,1]$ such that $L \cup P' \models_{tight} (\mathsf{BCD}|\mathsf{A})[l,u]$, where $L = \emptyset$ and $P' = \{(\mathsf{B}|\mathsf{A})[0,.2], (\mathsf{C}|\mathsf{AB})[0,.2]\}$. Then, we have:

$$
\mathcal{E} \;=\; \mathcal{E}_{(L,P')} \cup \{\mathsf{BCD}|\mathsf{A}\} \;=\; \{\mathsf{B}|\mathsf{A}, \mathsf{C}|\mathsf{AB}, \mathsf{BCD}|\mathsf{A}\}\,,
$$

and the set $R = R_L(\mathcal{E})$ is represented as follows:

$$
R \;\hat{=}\; \{\mathsf{A}\bar{\mathsf{B}}, \mathsf{AB}\bar{\mathsf{C}}, \mathsf{ABCD}, \mathsf{ABC}\bar{\mathsf{D}}\}\,.
$$

Thus, the system of linear constraints (3) is given by:

$$
\begin{aligned}
0.2\, y_{\mathsf{A}\bar{\mathsf{B}}} - 0.8\,(y_{\mathsf{AB}\bar{\mathsf{C}}} + y_{\mathsf{ABCD}} + y_{\mathsf{ABC}\bar{\mathsf{D}}}) \;&\geq\; 0\\
0.2\, y_{\mathsf{AB}\bar{\mathsf{C}}} - 0.8\,(y_{\mathsf{ABCD}} + y_{\mathsf{ABC}\bar{\mathsf{D}}}) \;&\geq\; 0\\
y_{\mathsf{A}\bar{\mathsf{B}}} + y_{\mathsf{AB}\bar{\mathsf{C}}} + y_{\mathsf{ABCD}} + y_{\mathsf{ABC}\bar{\mathsf{D}}} \;&=\; 1\\
y_{\mathsf{A}\bar{\mathsf{B}}}, y_{\mathsf{AB}\bar{\mathsf{C}}}, y_{\mathsf{ABCD}}, y_{\mathsf{ABC}\bar{\mathsf{D}}} \;&\geq\; 0\,.
\end{aligned}
\tag{5}
$$

Observe that (5) is solvable. By Theorem 4.3, it thus follows that $L \cup P'$ has a model Pr with $Pr(\mathsf{A}) > 0$. Hence, by Theorem 4.4, the requested l (resp., u) is given by the optimal value of the following linear program:

minimize (resp., maximize) y_{ABCD} subject to (5) . □

5 Techniques for Conjunctive Case

As we reduce the problems of deciding g-coherence and of computing tight g-coherent consequences to tasks in probabilistic logic, we can now apply all the techniques that have been developed there for an increased efficiency.

These techniques aim at reducing the exponential number of variables in the system of linear constraints (3).

In this section, we describe two such techniques for an increased efficiency that apply to the conjunctive case, namely (i) removing inactive logical and conditional constraints, and (ii) producing a reduced set of variables for the linear optimization problems in Theorems 4.3 and 4.4.

The key idea behind (i) is that some basic events can be assigned the probability zero, when deciding whether a given event has a positive probability, and when computing tight logical consequences. These basic events and all constraints in which they appear are then marked as *inactive*, and they are simply removed. The main idea behind (ii) is that we actually do not need the fine grain structure of $R_L(\mathcal{E}_{KB} \cup \{\alpha|\alpha\})$ (resp., $R_L(\mathcal{E}_{KB} \cup \{\beta|\alpha\})$), and thus some variables can be removed.

5.1 Preliminaries

An event ϕ is *conjunctive* iff it is either $\top$ or a conjunction of basic events. A conditional constraint $(\psi|\phi)[l,u]$ (resp., conditional event $\psi|\phi$) is *conjunctive* iff ψ is a conjunction of basic events, and ϕ is either $\top$ or a conjunction of basic events. A logical constraint $\psi \Leftarrow \phi$ is *Horn* (resp., *definite-Horn*) iff ψ is either $\bot$ or a basic event (resp., ψ is a basic event), and ϕ is either $\top$ or a conjunction of basic events. A probabilistic knowledge base $KB = (L, P)$ is *conjunctive* iff L is a finite set of Horn logical constraints, and P is a finite set of conjunctive conditional constraints.

5.2 Removing Inactive Constraints

We now describe a method in which some logical and conditional constraints are characterized as inactive and then removed. It is inspired by ideas from [28, 16]. Note that removing inactive constraints can be done in linear time.

For sets of Horn logical constraints L, we use L^+ to denote the set of all definite members of L. For sets of conjunctive conditional constraints P, denote by P^+ the set of all definite-Horn logical constraints $b \Leftarrow \phi$ such that $b \in \Phi$ occurs in the consequent of some $(\psi|\phi)[l,u] \in P$ with $l > 0$.

Let $KB = (L, P)$ be a conjunctive probabilistic knowledge base, and let α be a conjunctive event. A basic event $b \in \Phi$ is *active* w.r.t. KB and α, iff $L^+ \cup P^+ \cup \{\alpha\} \models b$. An event ϕ (resp., a logical or conditional constraint F) is *active* w.r.t. KB and α, iff all basic events in ϕ (resp., F) are active w.r.t. KB and α. An event (resp., a logical or conditional constraint) is *inactive* w.r.t. KB and α, iff it is not active w.r.t. KB and α. We define $A_\alpha(KB)$ as the probabilistic knowledge base (L_a, P_a) such that L_a (resp., P_a) denotes the set of all members of L (resp., P) that are active w.r.t. KB and α.

The following two theorems show that, as far as the tasks **PP** and **TLC** in Section 4.2 are concerned, we can remove all inactive logical and conditional constraints from KB. Roughly speaking, the main idea behind these results is that all inactive basic events can be assigned the probability zero, when solving **PP** and **TLC**. As a consequence, all inactive constraints then have some inactive basic events in their antecedent, and can thus simply be removed.

Theorem 5.1 *Let $KB = (L, P)$ be a conjunctive probabilistic knowledge base, and let α be a conjunctive event. Then, KB has a model Pr with $Pr(\alpha) > 0$ iff $A_\alpha(KB)$ has a model Pr with $Pr(\alpha) > 0$.*

Theorem 5.2 *Let $KB = (L, P)$ be a conjunctive probabilistic knowledge base, and $F = (\beta|\alpha)[l,u]$ be a conjunctive conditional constraint. Then,*

$$KB \models_{tight} F \text{ iff } A_{\alpha\wedge\beta}(KB) \models_{tight} F .$$

5.3 Reduced Set of Variables

In the conjunctive case, the number of variables in the systems of linear constraints for Theorems 4.3 and 4.4 can be directly reduced by a technique proposed in [29].

In the sequel, let $\mathcal{E}$ be a finite set of conjunctive conditional events. Let $\mathcal{C}$ be the following set of conjunctive events:

$$\mathcal{C} = \bigcup \{\{\psi \wedge \phi, \phi\} \mid \psi|\phi \in \mathcal{E}\}.$$

The operator S associates with each conjunctive event c the set of all contained basic events $b \in \Phi$. The operator T_L associates with each $B \subseteq \Phi$ the set of all $b \in \Phi$ such that $L \cup B \models b$. An element $c \in \mathcal{C}$ is *maximal* in $\mathcal{C}$ iff $L \cup \{c\} \not\models c'$ for all $c' \in \mathcal{C} \setminus \{c\}$. We use $max(\mathcal{C})$ to denote the set of all maximal elements in $\mathcal{C}$. Denote by $\mathcal{C}^\bullet$ the least set of subsets of Φ such that:

(i) If $c \in \mathcal{C}$, then $T_L(S(c)) \in \mathcal{C}^\bullet$.

(ii) If $c \in max(\mathcal{C})$, $C_1, C_2 \in \mathcal{C}^\bullet$, and $T_L(S(c)) \subseteq C_1, C_2$, then $T_L(C_1 \cup C_2) \in \mathcal{C}^\bullet$.

For each $C \in \mathcal{C}^\bullet$, let the mapping r_C associate with every

$\psi|\phi \in \mathcal{E}$ an element of $\{\psi \wedge \phi, \neg\psi \wedge \phi, \neg\phi\}$ as follows:

$$r_C(\psi|\phi) = \begin{cases} \psi \wedge \phi & \text{if } L \cup C \models \psi \text{ and } L \cup C \models \phi \\ \neg\psi \wedge \phi & \text{if } L \cup C \not\models \psi \text{ and } L \cup C \models \phi \\ \neg\phi & \text{if } L \cup C \not\models \phi. \end{cases}$$

We define $R^{\bullet}_L(\mathcal{E})$ as the set of all r_C such that (i) $C \in \mathcal{C}^{\bullet}$ and (ii) $L \cup \{r_C(\varepsilon) \mid \varepsilon \in \mathcal{E}\}$ is satisfiable.

The next two theorems show that we can use $R^{\bullet}_L$ instead of R_L for generating the variables in the systems of linear constraints for Theorems 4.3 and 4.4, respectively.

Theorem 5.3 *Let $KB = (L, P)$ be a conjunctive probabilistic knowledge base, and α be a conjunctive event. Let $R = R^{\bullet}_L(\mathcal{E}_{KB} \cup \{\alpha|\alpha\})$. Then, KB has a model Pr such that $Pr(\alpha) > 0$ iff the system of linear constraints (3) over the variables y_r $(r \in R)$ is solvable (see Fig. 3).*

Theorem 5.4 *Let $KB = (L, P)$ be a conjunctive probabilistic knowledge base, and α, β be conjunctive events. Let $R = R^{\bullet}_L(\mathcal{E}_{KB} \cup \{\beta|\alpha\})$. Suppose that KB has a model Pr such that $Pr(\alpha) > 0$. Then, l (resp., u) such that $KB \models_{tight} (\beta|\alpha)[l, u]$ is given by the optimal value of the following linear program over the variables y_r $(r \in R)$:*

$$\textit{minimize (resp., maximize)} \sum_{r \in R,\, r \models \beta \wedge \alpha} y_r \quad \textit{subject to } (3)\,.$$

We give an illustrative example.

Example 5.5 Consider again the probabilistic knowledge base $KB = (L, P)$ given in Example 4.2. Assume that we ask whether KB has a model Pr such that $Pr(\mathsf{A}) > 0$.

The set $\mathcal{E} = \mathcal{E}_{KB} \cup \{\mathsf{A}|\mathsf{A}\}$ is given as follows:

$$\mathcal{E} = \{\mathsf{J}|\mathsf{G}, \mathsf{G}|\mathsf{A}, \mathsf{G}|\mathsf{B}, \mathsf{G}|\mathsf{C}, \mathsf{J}|\mathsf{H}, \mathsf{J}|\mathsf{I}, \mathsf{I}|\mathsf{D}, \mathsf{I}|\mathsf{E}, \mathsf{I}|\mathsf{F}, \mathsf{A}|\mathsf{A}\}\,.$$

We observe that $R_L(\mathcal{E})$ contains 162 elements. The set $\mathcal{C}^{\bullet}$ contains only 10 elements and is given by:

$$\begin{aligned} \mathcal{C}^{\bullet} = \{&\{\mathsf{A}\}, \{\mathsf{B}\}, \{\mathsf{C}\}, \{\mathsf{D}\}, \{\mathsf{E}\}, \{\mathsf{F}\}, \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{G}\}, \\ &\{\mathsf{H}\}, \{\mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{I}\}, \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{G}, \mathsf{H}, \mathsf{I}, \mathsf{J}\}\}\,. \end{aligned}$$

Hence, also $R^{\bullet}_L(\mathcal{E})$ contains only 10 elements. More precisely, $R^{\bullet} = R^{\bullet}_L(\mathcal{E})$ is represented as follows:

$$\begin{aligned} R^{\bullet} \mathrel{\hat{=}} \{&\mathsf{A}\bar{\mathsf{B}}\bar{\mathsf{C}}\bar{\mathsf{D}}\bar{\mathsf{E}}\bar{\mathsf{F}}\bar{\mathsf{G}}\bar{\mathsf{H}}\bar{\mathsf{I}}, \bar{\mathsf{A}}\mathsf{B}\bar{\mathsf{C}}\bar{\mathsf{D}}\bar{\mathsf{E}}\bar{\mathsf{F}}\bar{\mathsf{G}}\bar{\mathsf{H}}\bar{\mathsf{I}}, \bar{\mathsf{A}}\bar{\mathsf{B}}\mathsf{C}\bar{\mathsf{D}}\bar{\mathsf{E}}\bar{\mathsf{F}}\bar{\mathsf{G}}\bar{\mathsf{H}}\bar{\mathsf{I}}, \\ &\bar{\mathsf{A}}\bar{\mathsf{B}}\bar{\mathsf{C}}\mathsf{D}\bar{\mathsf{E}}\bar{\mathsf{F}}\bar{\mathsf{G}}\bar{\mathsf{H}}\bar{\mathsf{I}}, \bar{\mathsf{A}}\bar{\mathsf{B}}\bar{\mathsf{C}}\bar{\mathsf{D}}\mathsf{E}\bar{\mathsf{F}}\bar{\mathsf{G}}\bar{\mathsf{H}}\bar{\mathsf{I}}, \bar{\mathsf{A}}\bar{\mathsf{B}}\bar{\mathsf{C}}\bar{\mathsf{D}}\bar{\mathsf{E}}\mathsf{F}\bar{\mathsf{G}}\bar{\mathsf{H}}\bar{\mathsf{I}}, \\ &\mathsf{A}\mathsf{B}\mathsf{C}\bar{\mathsf{D}}\bar{\mathsf{E}}\bar{\mathsf{F}}\mathsf{G}\bar{\mathsf{H}}\bar{\mathsf{I}}\,\bar{\mathsf{J}}, \bar{\mathsf{A}}\bar{\mathsf{B}}\bar{\mathsf{C}}\bar{\mathsf{D}}\bar{\mathsf{E}}\bar{\mathsf{F}}\bar{\mathsf{G}}\mathsf{H}\bar{\mathsf{I}}\,\bar{\mathsf{J}}, \\ &\bar{\mathsf{A}}\bar{\mathsf{B}}\bar{\mathsf{C}}\mathsf{D}\mathsf{E}\mathsf{F}\bar{\mathsf{G}}\bar{\mathsf{H}}\mathsf{I}\bar{\mathsf{J}}, \mathsf{A}\mathsf{B}\mathsf{C}\mathsf{D}\mathsf{E}\mathsf{F}\mathsf{G}\mathsf{H}\mathsf{I}\mathsf{J}\}\,. \end{aligned}$$

Thus, the number of variables reduces from 162 to 10. □

5.4 Random Gain

Interestingly, the technique of removing variables in Section 5.3 can be expressed in terms of random gain.

Let $KB = (L, P)$ be a probabilistic knowledge base with $P = \{(\psi_1|\phi_1)[l_1, u_1], \ldots, (\psi_n|\phi_n)[l_n, u_n]\}$ and $n \geq 1$. Let $R = R_L(\mathcal{E}_{KB})$. Then, for every $r \in R$, the corresponding random gain g_r can be represented as follows:

$$g_r = \sum_{i=1}^{n} s_i(p_{ri} - l_i) - \sigma_i(q_{ri} - u_i)\,, \tag{6}$$

where $s_i \geq 0$ and $\sigma_i \geq 0$ are real numbers, and p_{ri} and q_{ri} are defined by (2), for all $r \in R$ and $i \in \{1, \ldots, n\}$.

The following theorem now shows that $R^{\bullet}_L(\mathcal{E}_{KB})$ can be characterized using the random gain.

Theorem 5.6 *Let $KB = (L, P)$ be a conjunctive probabilistic knowledge base where $P = \{(\psi_1|\phi_1)[l_1, u_1], \ldots, (\psi_n|\phi_n)[l_n, u_n]\}$ and $n \geq 1$. Let $R = R_L(\mathcal{E}_{KB})$, $R^{\bullet} = R^{\bullet}_L(\mathcal{E}_{KB})$, and $r \in R$. Then, the following holds:*

(a) If $r \in R^{\bullet}$, then no $S \subseteq R$ exists such that $|S| \geq 2$ and $g_r = \sum_{a \in S} g_a$.

(b) If $r \in R \setminus R^{\bullet}$, then $g_r = \sum_{a \in S} g_a$ for some $S \subseteq R^{\bullet}$ with $|S| \geq 2$.

6 Summary and Outlook

We showed that the notions of g-coherence and of g-coherent entailment can be expressed by combining notions in model-theoretic probabilistic logic with concepts from default reasoning. Using these results, we analyzed the complexity of probabilistic reasoning under coherence. Moreover, we gave new algorithms for deciding g-coherence and for computing tight g-coherent intervals, which reduce these tasks to standard reasoning tasks in model-theoretic probabilistic logic. Thus, efficient methods for model-theoretic probabilistic reasoning can immediately be applied for probabilistic reasoning under coherence. We then described two such techniques for the conjunctive case.

We remark that removing inactive constraints as described in Section 5.2 can be applied in combination with column generation techniques. We expect that the same applies to the reduced set of variables described in Section 5.3. Exploring this aspect in depth is a subject of future research.

Another interesting topic of future research is to further explore the reduced set of variables described in Section 5.3. In particular, it has been shown in [5] that further variables can be removed in coherence-based probabilistic reasoning by a refined characterization based on random gain. We expect that this result carries over to model-theoretic probabilistic reasoning.

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References

[1] S. Amarger, D. Dubois, and H. Prade. Constraint propagation with imprecise conditional probabilities. In *Proceedings UAI-91*, pages 26–34. Morgan Kaufmann, 1991.

[2] V. Biazzo and A. Gilio. A generalization of the fundamental theorem of de Finetti for imprecise conditional probability assessments. *Int. J. Approx. Reasoning*, 24(2–3):251–272, 2000.

[3] V. Biazzo, A. Gilio, T. Lukasiewicz, and G. Sanfilippo. Probabilistic logic under coherence, model-theoretic probabilistic logic, and default reasoning. Technical Report INFSYS RR-1843-01-03, Institut für Informationssysteme, TU Wien, 2001. Available at `ftp://ftp.kr.tuwien.ac.at/pub/tr/rr0103.ps.gz`.

[4] V. Biazzo, A. Gilio, T. Lukasiewicz, and G. Sanfilippo. Probabilistic logic under coherence: Complexity and algorithms. Technical Report INFSYS RR-1843-01-04, Institut für Informationssysteme, TU Wien, 2001. Available at `ftp://ftp.kr.tuwien.ac.at/pub/tr/rr0104.ps.gz`.

[5] V. Biazzo, A. Gilio, and G. Sanfilippo. Computational aspects in checking of coherence and propagation of conditional probability bounds. In *Proceedings WUPES-2000*, pages 1–13, Jindřichův Hradec, Czech Republic, 2000.

[6] V. Biazzo, A. Gilio, and G. Sanfilippo. Efficient checking of coherence and propagation of imprecise probability assessments. In *Proceedings IPMU-2000*, pages 1973–1976, Madrid, Spain, 2000.

[7] G. Boole. *An Investigation of the Laws of Thought, on which are Founded the Mathematical Theories of Logic and Probabilities*. Walton and Maberley, London, 1854. (reprint: Dover Publications, New York, 1958).

[8] A. Capotorti, L. Galli, and B. Vantaggi. How to use strong local coherence in an inferential process based on upper-lower probabilities. In *Proceedings WUPES-2000*, pages 14–28, Jindřichův Hradec, Czech Republic, 2000.

[9] A. Capotorti and B. Vantaggi. An algorithm for coherent conditional probability assessments. In *Proceedings IV Congresso Nazionale SIMAI*, volume 2, pages 144–148, Giardini Naxos (Messina), Italy, 1998.

[10] G. Coletti. Coherent numerical and ordinal probabilistic assessments. *IEEE Transactions on Systems, Man, and Cybernetics*, 24(12):1747–1754, 1994.

[11] G. Coletti and R. Scozzafava. Characterization of coherent conditional probabilities as a tool for their assessment and extension. *Journal of Uncertainty, Fuzziness and Knowledge-based Systems*, 4(2):103–127, 1996.

[12] G. Coletti and R. Scozzafava. Exploiting zero probabilities. In *Proceedings EUFIT-97*, pages 1499–1503, Aachen, Germany, 1997.

[13] G. Coletti and R. Scozzafava. Coherent upper and lower Bayesian updating. In *Proceedings ISIPTA-99*, pages 101–110, Ghent, Belgium, 1999.

[14] G. Coletti and R. Scozzafava. Conditioning and inference in intelligent systems. *Soft Computing*, 3(3):118–130, 1999.

[15] D. Dubois, H. Prade, and J.-M. Touscas. Inference with imprecise numerical quantifiers. In Z. W. Ras and M. Zemankova, editors, *Intelligent Systems*, chapter 3, pages 53–72. Ellis Horwood, 1990.

[16] T. Eiter and T. Lukasiewicz. Default reasoning from conditional knowledge bases: Complexity and tractable cases. *Artif. Intell.*, 124(2):169–241, 2000.

[17] R. Fagin, J. Y. Halpern, and N. Megiddo. A logic for reasoning about probabilities. *Inf. Comput.*, 87:78–128, 1990.

[18] A. M. Frisch and P. Haddawy. Anytime deduction for probabilistic logic. *Artif. Intell.*, 69:93–122, 1994.

[19] D. M. Gabbay and P. Smets, editors. *Handbook on Defeasible Reasoning and Uncertainty Management Systems*. Kluwer Academic, Dordrecht, Netherlands, 1998.

[20] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. Freeman, New York, 1979.

[21] G. Georgakopoulos, D. Kavvadias, and C. Papadimitriou. Probabilistic satisfiability. *J. Complexity*, 4(1):1–11, 1988.

[22] A. Gilio. Probabilistic consistency of conditional probability bounds. In *Advances in Intelligent Computing*, *LNCS* 945, pages 200–209. Springer, 1995.

[23] A. Gilio. Precise propagation of upper and lower probability bounds in system P. In *Proceedings of the 8th International Workshop on Non-monotonic Reasoning*, Breckenridge, Colorado, USA, 2000.

[24] A. Gilio and R. Scozzafava. Conditional events in probability assessment and revision. *IEEE Transactions on Systems, Man, and Cybernetics*, 24(12):1741–1746, 1994.

[25] M. Goldszmidt and J. Pearl. On the consistency of defeasible databases. *Artif. Intell.*, 52(2):121–149, 1991.

[26] P. Hansen, B. Jaumard, G.-B. D. Nguetsé, and M. P. de Aragão. Models and algorithms for probabilistic and Bayesian logic. In *Proceedings IJCAI-95*, pages 1862–1868, 1995.

[27] B. Jaumard, P. Hansen, and M. P. de Aragão. Column generation methods for probabilistic logic. *ORSA J. Comput.*, 3:135–147, 1991.

[28] T. Lukasiewicz. Probabilistic logic programming with conditional constraints. *ACM Transactions on Computational Logic*, 2001. To appear.

[29] T. Lukasiewicz. Efficient global probabilistic deduction from taxonomic and probabilistic knowledge-bases over conjunctive events. In *Proceedings CIKM-97*, pages 75–82. ACM Press, 1997.

[30] T. Lukasiewicz. Local probabilistic deduction from taxonomic and probabilistic knowledge-bases over conjunctive events. *Int. J. Approx. Reasoning*, 21(1):23–61, 1999.

[31] T. Lukasiewicz. Probabilistic deduction with conditional constraints over basic events. *Journal of Artificial Intelligence Research*, 10:199–241, 1999.

[32] N. J. Nilsson. Probabilistic logic. *Artif. Intell.*, 28:71–88, 1986.

[33] R. Pelessoni and P. Vicig. A consistency problem for imprecise conditional probability assessments. In *Proceedings IPMU-98*, pages 1478–1485, Paris, France, 1998.

[34] R. Scozzafava. Subjective conditional probability and coherence principles for handling partial information. *Mathware & Soft Computing*, 3(1):183–192, 1996.

[35] P. Vicig. An algorithm for imprecise conditional probability assessments in expert systems. In *Proceedings IPMU-96*, pages 61–66, Granada, Spain, 1996.

[36] P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.