

## A Multiperiod Binomial Model for Pricing Options in an Uncertain World

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### Abstract

The aim of this paper is to price an option in a multiperiod binomial model, when there is uncertainty on the states of the world at each node of the tree. As a consequence, also the stock price at each state takes imprecise values. Possibility distributions are used to handle this type of problems. The pricing methodology is still based on a risk neutral valuation approach, but, as a consequence of the uncertainty on the two jumps of the stock, we obtain weighted intervals for risk-neutral probabilities. The distinctive feature of our model is that it tracks back the arising of these probability intervals to the imprecision of the value of the stock price in the up and down states. This paper provides a generalization of the standard binomial option pricing model. We obtain an expected value interval for the option price within which it is possible to find a crisp representative value and an index of the uncertainty present in the model.

**Keywords.** Evidence theory, Fuzzy sets, Options, Pricing.

### 1 Introduction

The aim of this paper is to price an option in a multiperiod binomial model, when there is uncertainty on the states of the world at each node of the tree. The starting point is the one period model, where the two states of the world, namely state up, where the market is “bullish” and state down, where the market is “bearish”, are vague. As a consequence, also the stock price at each state takes imprecise values.

In a standard one period binomial model, we are required to give two crisp values, one for each state, for the stock price movement in the next time period. This problem boils down to the estimation of the volatility of the underlying asset, that is an unobservable parameter. Sometimes it is hard to give precise estimates for the volatility and in turn for the stock price movements in the future. Possibility distributions are used to handle this type of problems.

The pricing methodology is still based on a risk neutral valuation approach, but, as a consequence of the vagueness in the two jumps of the stock, we obtain weighted intervals of risk-neutral probabilities that closely resemble the belief and plausibility measures of Evidence theory [5]. These measures are obtained by replacing the additivity requirement by superadditivity or subadditivity respectively. The dual relationship between the two types of measures ensures that given a measure of either of the two types, it induces a unique measure of the other type. Since belief measures are always smaller than or equal to the corresponding plausibility measures, they may be seen as lower and upper probabilities respectively.

Expected values can be computed under these measures resulting in an expected value interval. For applications to the pricing problem, this feature represents a drawback given that one needs some additional criterion in order to get a single price. Cherubini and Della Lunga [2] applied a particular class of fuzzy measures [13], the  $g$ -lambda measures, to the asset pricing problem and found intervals for the derivative price. In this paper, by modelling the stock price in each state of the world as a fuzzy number we obtain a possibility distribution on the risk neutral probability, i.e. a weighted interval of probability. This, by contrast with the theory of evidence, implies a main advantage for pricing problems. In fact, by computing the option price under this measure we get a weighted expected value interval for the price and thus we are able to determine a “most likely” value of the option within the interval.

Moreover, it may be convenient to synthesise the option price interval in one crisp constant that summarises all the information contained. This process is called defuzzification procedure. We can also get an index of the fuzziness present in the option price, that tells us the degree of imprecision intrinsic in the model.

The plan of the paper is the following. In section 2 we describe the opacity in the possible jumps in stock prices by means of fuzzy triangular numbers. In section 3 we present the one period model set up, we derive the risk-

neutral probabilities and analyse their main characteristics. In section 4 we describe the payoff of options and discuss their pricing in this framework. In section 5 and 6 we extend our result to a multiperiod setting. In section 7 we propose a method for finding the scalar that best approximate the call price and an index for the fuzziness present in the model. The last section concludes. Appendix 1 and 2 report the properties of the risk-neutral probabilities and of the call price respectively.

## 2 The fuzzy binomial tree

In order to introduce our pricing methodology we first set up a one period model, with  $t \in [0,1]$ , characterised as follows. We assume that the two states of the world at  $t=1$  are uncertain: say that in state up the market is “bullish” and in state down the market is “bearish”. As a consequence the price of the underlying at  $t=1$  takes only two possible values: given the current value  $P_0$  it may either jump up or down with an exogenously given probability  $p$  and  $(1-p) p \in [0,1]$ .

In a standard one period binomial model, we are required to give two precise values for the increase or decrease in the stock price in the next time period. Let  $u$  and  $d$  be the up and down crisp jump factors respectively, the standard methodology [4] leads to set  $u = e^{s\sqrt{\Delta t}}$ ,  $d = e^{-s\sqrt{\Delta t}}$ , where  $\sigma$  is the volatility of the underlying asset and  $\Delta t$  is length of the time period.

There are different methods for estimating volatility either from historical data, or from option prices [9]. Sometimes it is hard to give a precise estimate of the volatility of the underlying asset and it may be convenient to let it take interval values. Moreover, it may be the case that not all the members of the interval have the same reliability, as central members are more possible than the ones near the borders. This is exactly the idea behind our model, but instead of modelling volatility as a fuzzy quantity, we directly model the up and down jumps of the stock price.

The fuzziness present in the model is due to the fact that there is uncertainty about the exact increase or decrease in the stock price. We thus have two possibility distributions: one for the increase and one for the decrease of the stock price, as illustrated in Figure 1. The up and down jump factors,  $u$  and  $d$  respectively, are represented by two triangular fuzzy numbers. More simply, for each state, we can write:  $I=(i_1, i_2, i_3)$ ,  $i=\{d, u\}$ , where  $i_1$  is the minimum possible value,  $i_3$  the maximum, and  $i_2$  the most possible. The possibility distribution is induced by the characteristic function of the fuzzy set. Alternatively, we can write a triangular fuzzy number in

terms of its  $\mathbf{a}$ -cuts (or confidence intervals) by the following formula:

$$i(\mathbf{a})=[i_1(\mathbf{a}), i_3(\mathbf{a})]=[i_1 + \mathbf{a}(i_2 - i_1), i_3 - \mathbf{a}(i_3 - i_2)]$$

where  $\mathbf{a}$  is the level of confidence,  $\mathbf{a} \in \tilde{I}[0,1]$ ,  $i=\{d, u\}$ . This representation will be useful to do some algebra with fuzzy numbers. We can estimate from market data the parameters that are input to our model. Alternatively, we can use the standard methodology and set

$$u_1 = e^{s_1\sqrt{\Delta t}}, u_2 = e^{s_2\sqrt{\Delta t}}, u_3 = e^{s_3\sqrt{\Delta t}}$$

$$d_1 = 1/u_3, d_2 = 1/u_2, d_3 = 1/u_1$$

where  $\sigma_1$  and  $\sigma_3$  are the lower and upper bounds for volatility respectively and  $\sigma_2$  is the most possible value.

Triangular fuzzy numbers are used to model the two possibility distributions. Among all the different types of numbers, the choice of using triangular numbers is made for the sake of simplicity, since assuming more complicated shapes may increase the computational complexity without substantially affecting the significance of the results. Even if the price of the stock is constrained to move up or down in discrete ticks, there is still uncertainty about the number of ticks it will increase or decrease. For most stocks, daily price movement limits are specified by the exchange, these limits may be used as lower or upper bounds for the support of the decrease or increase in the stock price respectively.

This way of modelling the imprecision has an intuitive appeal. In the standard binomial model we say that the stock price at time one may jump up or down to two exactly given values, for example being 100 the stock price at time zero, it may jump to exactly 110 or exactly 90 at time one. In our model we just say that if the market is bullish the stock price increases and if the market is bearish it will decrease, whereby the amount of the change is imprecisely given. Following this example the stock price in our model is allowed to jump to “approximately 110” or to “more or less 90”. In other words, we fix the peak value of the fuzzy number equal to the crisp value in the standard binomial case and we allow the nearby prices to have some degree of possibility.

## 3 The risk-neutral probabilities interval

The aim of this section is to derive the risk-neutral probabilities in order to price a call written on a stock. Let us consider a one-period model where  $t \in [0,1]$  is time, the two basic securities are: the money market account, and the risky stock. The money market account, is worth one at  $t=0$  and its value at  $t=1$  is  $1+r$ , where  $r$  is the risk-free interest rate.

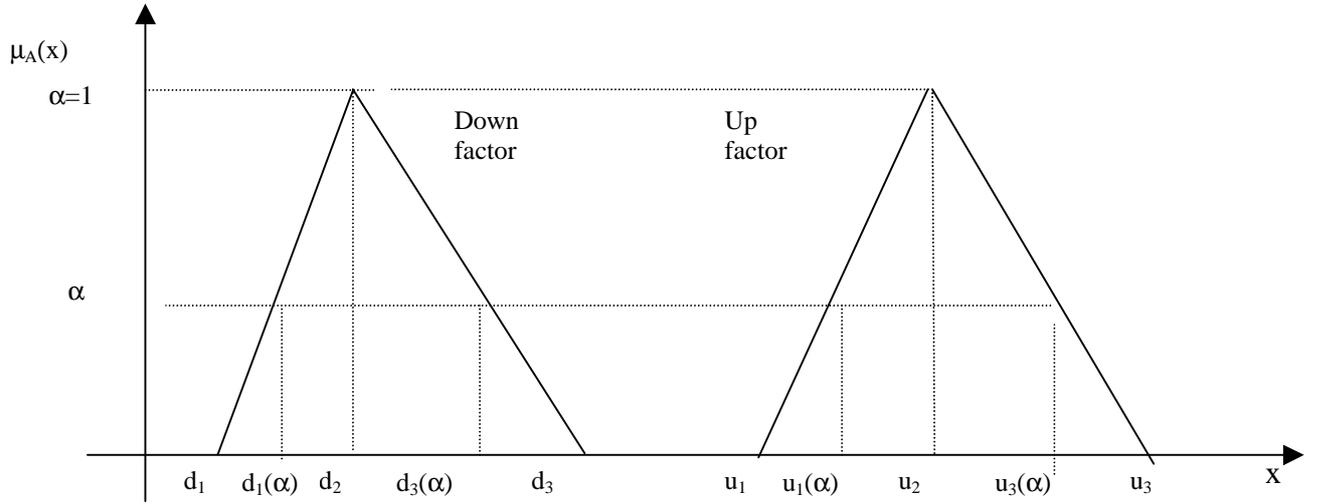


Figure 1: The two possible jump factors of the underlying at t=1.

The stock price at time zero,  $P_0$ , is observable, while its price at time one, is obtained multiplying  $P_0$  by the jump factors introduced in the previous section.

The following assumptions are made:

A1) All investors have homogeneous beliefs, they all agree on the shape of the triangular numbers used to represent the up and down jump factors.

A2) Markets are frictionless i.e. markets have no transaction costs, no taxes, no restrictions on short sales and asset are infinitely divisible.

A3) Every investor acts as a price taker.

A4) Interest rates are positive. The interest rate is equal to  $r > 0$  percent per unit time.

A5) No arbitrage opportunities are allowed. This condition is expressed by the following formula:

$$d_3 < (1+r) < u_1 \quad (1)$$

A6) The market is complete. A market is complete if there are as many independent (not replicable) instruments as states. In our case we have two instruments (the fuzzy stock and the money market account) and two states of the world (bullish and bearish).

Under this set of assumptions, we can apply the risk neutral valuation principle. In a risk neutral world, all individuals are assumed to be indifferent to risk. It follows that the expected return on every security is the risk free rate. The real probabilities do not play any role in the determination of the risk neutral probabilities that depend only on the risk free rate, and on the magnitude of the up and down jumps of the stock [9]. The standard methodology for deriving the risk-neutral probabilities yields to the system:

$$\begin{bmatrix} 1 & 1 \\ \frac{P_0 d}{1+r} & \frac{P_0 u}{1+r} \end{bmatrix} \begin{bmatrix} p_d \\ p_u \end{bmatrix} = \begin{bmatrix} 1 \\ P_0 \end{bmatrix}$$

or alternatively:

$$\begin{cases} p_d + p_u = 1 \\ \frac{d}{1+r} p_d + \frac{u}{1+r} p_u = 1 \end{cases}$$

By writing u and d in terms of  $\alpha$ -cut, we get:

$$\begin{cases} p_d + p_u = 1 \\ \frac{[d_1^a, d_3^a]}{1+r} p_d + \frac{[u_1^a, u_3^a]}{1+r} p_u = [1, 1] \end{cases}$$

where if A is a real number, then  $A(\mathbf{a}) = A \forall \mathbf{a}$ .

This system may be split into the following two [1],[6]:

$$\begin{cases} p_d + p_u = 1 \\ \frac{d_1 + \mathbf{a}(d_2 - d_1)}{1+r} p_d + \frac{u_1 + \mathbf{a}(u_2 - u_1)}{1+r} p_u = 1 \end{cases} \quad (2)$$

$$\begin{cases} p_d + p_u = 1 \\ \frac{d_3 - \mathbf{a}(d_3 - d_2)}{1+r} p_d + \frac{u_3 - \mathbf{a}(u_3 - u_2)}{1+r} p_u = 1 \end{cases} \quad (3)$$

Solving system (2) yields:

$$\begin{cases} p_u = \frac{(1+r) - d_1 - \mathbf{a}(d_2 - d_1)}{u_1 - d_1 + \mathbf{a}(u_2 - u_1 - d_2 + d_1)} \\ p_d = \frac{u_1 + \mathbf{a}(u_2 - u_1) - (1+r)}{u_1 - d_1 + \mathbf{a}(u_2 - u_1 - d_2 + d_1)} \end{cases}$$

solving system (3) yields:

$$\begin{cases} p_u = \frac{(1+r) - d_3 + \mathbf{a}(d_3 - d_2)}{u_3 - d_3 - \mathbf{a}(u_3 - u_2 - d_3 + d_2)} \\ p_d = \frac{u_3 - \mathbf{a}(u_3 - u_2) - (1+r)}{u_3 - d_3 - \mathbf{a}(u_3 - u_2 - d_3 + d_2)} \end{cases}$$

The two solutions represent the bounds of the intervals of probabilities that are respectively:

$$\begin{aligned} p_u = [\underline{p}_u, \bar{p}_u] &= \left[ \frac{(1+r) - d_3 + \mathbf{a}(d_3 - d_2)}{u_3 - d_3 - \mathbf{a}(u_3 - u_2 - d_3 + d_2)}, \frac{(1+r) - d_1 - \mathbf{a}(d_2 - d_1)}{u_1 - d_1 + \mathbf{a}(u_2 - u_1 - d_2 + d_1)} \right] \\ p_d = [\underline{p}_d, \bar{p}_d] &= \left[ \frac{u_1 + \mathbf{a}(u_2 - u_1) - (1+r)}{u_1 - d_1 + \mathbf{a}(u_2 - u_1 - d_2 + d_1)}, \frac{u_3 - \mathbf{a}(u_3 - u_2) - (1+r)}{u_3 - d_3 - \mathbf{a}(u_3 - u_2 - d_3 + d_2)} \right] \end{aligned} \quad (4)$$

It is easy to check that the following duality relations hold:  $\bar{p}_i + \underline{p}_i = 1$  and  $\bar{p}_d + \underline{p}_u = 1$ . To draw a comparison with Evidence Theory, we indeed have two measures,  $\bar{p}_i$  and  $\underline{p}_i$ , with  $i=d,u$ , where  $\bar{p}_i$  is the dual measure of  $\underline{p}_i$ .

It is interesting to observe that, differently from the standard binomial option pricing model, we obtain risk neutral probability intervals instead of point estimates. This is clearly a consequence of our assumption on the stock price. The risk neutral probability intervals arise from the opacity of the stock price at  $t=1$ , even if the real probabilities of the stock price jumps are crisp and known in advance. This is a consequence of the risk neutral valuation paradigm, that states that if the market is complete and there are no arbitrage opportunities, then the real probabilities involved do not play any role in the pricing problem. The risk neutral probability measure depends only on the amount of decrease or increase in the stock price. If the jump factors are crisp numbers, then we are back to the standard binomial model, with crisp risk neutral probabilities. In this setting, the intervals of risk-neutral probabilities are weighted. This is a very important feature of our pricing model, since it allows us to find a weighted expected value interval for the call price as is shown in the following sections.

In order to determine the shape of the two probabilities, we compute their value at  $\alpha=0$  and  $\alpha=1$  and then we analyse their behaviour as  $\alpha$  varies (proofs are in Appendix 1).

If  $\alpha=0$  then:

$$\begin{aligned} p_u &= \left[ \frac{(1+r) - d_3}{u_3 - d_3}, \frac{(1+r) - d_1}{u_1 - d_1} \right] \\ p_d &= \left[ \frac{u_1 - (1+r)}{u_1 - d_1}, \frac{u_3 - (1+r)}{u_3 - d_3} \right] \end{aligned} \quad (5)$$

If  $\alpha=1$  then:

$$\begin{aligned} p_u &= \frac{(1+r) - d_2}{u_2 - d_2} \\ p_d &= \frac{u_2 - (1+r)}{u_2 - d_2} \end{aligned} \quad (6)$$

It is easy to show that the derivative with respect to  $\alpha$  is positive for both the left bounds and is negative for both the right bounds of the probabilities. This means that the bounds are getting narrower as  $\alpha$  increases. In particular, if  $\alpha=1$ , these bounds collapse in only one point. If  $\alpha=1$ , the stock price in each state assume only one value and as a consequence we find a unique risk neutral probability measure. Thus our model can be seen as a generalisation of the standard binomial option pricing model as the latter is a special case (if  $\alpha=1$ ) of the former.

By inspection of  $\underline{p}_u$ , it is easy to prove that its first derivative is positive and that the second derivative is positive if  $u_3 - u_2 > d_3 - d_2$ ; in the opposite case it is negative. Note that if  $u_3 - u_2 = d_3 - d_2$  then  $\underline{p}_u$  is linear in  $\alpha$ . Analogously it is easy to prove that the first derivative of  $\bar{p}_u$  is negative and that the second derivative is positive if  $u_2 - u_1 > d_2 - d_1$ ; in the opposite case it is negative. Note that if  $u_2 - u_1 = d_2 - d_1$  then  $\bar{p}_u$  is linear in  $\alpha$ . As for  $\underline{p}_d$ , we can prove that its first derivative is positive and that the second derivative is negative if  $u_2 - u_1 > d_2 - d_1$ ; in the opposite case it is positive. Note that if  $u_3 - u_2 = d_3 - d_2$  then  $\underline{p}_d$  is linear in  $\alpha$ . Analogously for  $\bar{p}_d$ , we can prove that its first derivative is negative and that the second derivative is negative if  $u_3 - u_2 > d_3 - d_2$ ; in the opposite case it is positive. Note that if  $u_3 - u_2 = d_3 - d_2$  then  $\bar{p}_d$  is linear in  $\alpha$ .

In sum, depending on the relative positions of  $u_1, u_2, u_3, d_1, d_2, d_3$  we can have different shapes for  $p_u$  and  $p_d$  as illustrated in Tables 1 and 2. The graphs, that are just possible outcomes, show how the probability intervals shrink with  $\alpha$ . In fact for  $\alpha=1$  each of the risk-neutral probabilities assumes a single value. Note that the two bounds and the most possible value are determined by equations (5) and (6). Note that in Tables 1 and 2 are not reported, for reasons of space, the cases in which we have  $u_2 - d_2 = u_1 - d_1$  or  $u_3 - d_3 = u_2 - d_2$ . It is easy to verify that if  $u_2 - d_2 = u_1 - d_1$  then both  $\bar{p}_u$  and  $\underline{p}_d$  are linear in  $\alpha$ ; if  $u_3 - d_3 = u_2 - d_2$  then both  $\underline{p}_u$  and  $\bar{p}_d$  are linear in  $\alpha$ .

As a special case we examine what happens if both the triangular fuzzy numbers that represent the up and down jump factors are symmetric and equally widespread, i.e. if  $u_3 - u_2 = u_2 - u_1 = d_3 - d_2 = d_2 - d_1 = k$ ,  $(7)$   $k$  being the left or right spread. Note that this implies also:  $u_3 - d_3 = u_2 - d_2 = u_1 - d_1 = h$ . In this case both  $p_u$  and  $p_d$  are linear in  $\alpha$ , i.e. they are triangular fuzzy numbers:

$$p_u = [\underline{p}_u, \bar{p}_u] = \left[ \frac{(1+r) - d_3 + \mathbf{a}k}{h}, \frac{(1+r) - d_1 - \mathbf{a}k}{h} \right] \quad (8)$$

$$p_d = [\underline{p}_d, \bar{p}_d] = \left[ \frac{u_1 + \mathbf{a}k - (1+r)}{h}, \frac{u_3 - \mathbf{a}k - (1+r)}{h} \right]$$

As we can see from Tables 1 and 2, the shape of the artificial probabilities depends on the shape of the up and down jump factors. In particular, let us examine what happens if we hold fixed the two peaks,  $d_2$  and  $u_2$  of the two jump factors. In Table 1 we can see that if the distribution of the mass of the up factor is closer to the down jump factor and viceversa, i.e. the up and down jump factors are less distinct, then both  $p_u$  and  $p_d$  are closer to a crisp number. The opposite holds if the up and down jump factors are more distinct, i.e. both  $p_u$  and  $p_d$  are more vague. In Table 2 we have two intermediate situations: if the up factor is wider than the down factor, then the left part of  $p_u$  is more vague than the right part and the opposite holds for  $p_d$ . If the down jump factor is wider than the up factor then the left part of  $p_u$  is more vague than the right part and the opposite holds for  $p_d$ .

#### 4 The pricing of an option

In this section, we use the risk-neutral probabilities obtained in the previous section in order to price an option. As we are in a one period model, it makes no sense to distinguish between American and European options. At the maturity date, a call option has a positive value if the price of the underlying is greater than the exercise price; in the opposite case it remains unexercised and has zero value. The payoff of a call option depends on the price of the underlying asset. The stock price at  $t=1$  is given by either  $P_0d$  or  $P_0u$ . Since  $u$  and  $d$  are triangular fuzzy numbers, it follows that the stock price at  $t=1$  in each state is represented by a triangular fuzzy number.

To make an option an interesting contract we assume that the strike price is between the highest value of the stock in state down and the lowest value of the stock in state up:  $P_0d_3 \leq X \leq P_0u_1$ . We denote the call payoff in state “up” with  $C(u)$  and in state down with  $C(d)$ . It follows that  $C(d) = 0$  and  $C(u) = (P_0u - X)$ . Applying the algebra of fuzzy numbers we obtain the call payoff, which is still a triangular fuzzy number equal to:  $C(u) = (P_0u_1 - X, P_0u_2 - X, P_0u_3 - X)$ . We now determine the call price  $C_0$  by means of the risk neutral valuation approach, as follows:

$$C_0 = \frac{1}{1+r} \hat{E}[C_1] = \frac{1}{1+r} [p_d * C(d) + p_u * C(u)]$$

where  $\hat{E}$  stands for expectation under the risk-neutral probabilities and  $C_1$  is the payoff of the call at  $t=1$ . Since the call has zero payoff in the down state, the option pricing formula simplifies to:

$$C_0 = \frac{1}{1+r} [p_u * C(u)].$$

Using  $\alpha$ -cuts in the multiplication between fuzzy numbers, as we have an interval for  $p_u$ :

$$p_u = [\underline{p}_u, \bar{p}_u] = \left[ \frac{(1+r) - d_3 + \mathbf{a}(d_3 - d_2)}{u_3 - d_3 - \mathbf{a}(u_3 - u_2 - d_3 + d_2)}, \frac{(1+r) - d_1 - \mathbf{a}(d_2 - d_1)}{u_1 - d_1 + \mathbf{a}(u_2 - u_1 - d_2 + d_1)} \right]$$

we also have an interval for the call prices.

$$C_0 = [\underline{C}_0, \bar{C}_0] = \left[ \frac{P_0u_1 - X + \mathbf{a}P_0(u_2 - u_1) * \frac{(1+r) - d_3 + \mathbf{a}(d_3 - d_2)}{u_3 - d_3 - \mathbf{a}(u_3 - u_2 - d_3 + d_2)}}{1+r}, \frac{P_0u_3 - X - \mathbf{a}P_0(u_3 - u_2) * \frac{(1+r) - d_1 - \mathbf{a}(d_2 - d_1)}{u_1 - d_1 + \mathbf{a}(u_2 - u_1 - d_2 + d_1)}}{1+r} \right] \quad (9)$$

It is easy to prove that as  $\alpha$  increases the call option interval of prices shrinks (for the proof see Appendix 2). It follows that if  $\alpha=0$  the price interval is the largest:

$$C_0 = \frac{1}{1+r} \left[ (P_0u_1 - X) * \frac{(1+r) - d_3}{u_3 - d_3}, (P_0u_3 - X) * \frac{(1+r) - d_1}{u_1 - d_1} \right]. \quad (10)$$

If  $\alpha=1$ , the interval collapses into one single value:

$$C_0 = \frac{P_0u_2 - X}{1+r} * \frac{(1+r) - d_2}{u_2 - d_2}. \quad (11)$$

which is the same result as in the standard binomial option pricing model.

By contrast with models, where the case of non additive measures implies simple price intervals, we get a weighted expected value interval for the call price. This is clearly a very important feature for financial applications since it enables us to determine the most possible outcome of the call price. It is also interesting to observe that the “most likely value” of the call is the one that we would have obtained in a standard binomial option pricing model.

Analysing the shape of the call option price we note that the left part:

$$\underline{C}_0 = \frac{P_0u_1 - X + \mathbf{a}P_0(u_2 - u_1) * \frac{(1+r) - d_3 + \mathbf{a}(d_3 - d_2)}{u_3 - \mathbf{a}(u_3 - u_2) - d_3 + \mathbf{a}(d_3 - d_2)}}{1+r}$$

is increasing in  $\alpha$  and is concave or convex depending on the sign of the quantity

$(u_3 - d_3)(P_0u_2 - X) - (u_2 - d_2)(P_0u_1 - X)$  as illustrated in Table 2 (for the proofs see Appendix 2). Note that if  $(u_3 - d_3)(P_0u_2 - X) = (u_2 - d_2)(P_0u_1 - X)$ , then is linear in  $\alpha$

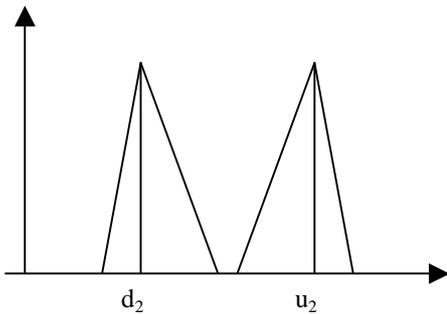
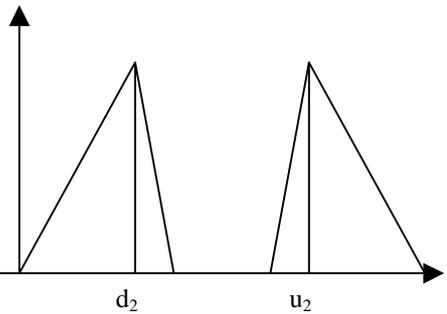
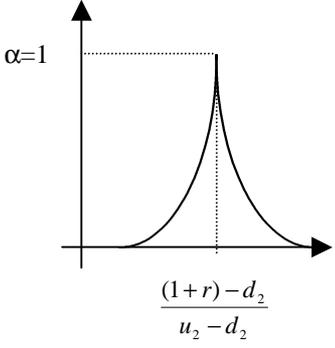
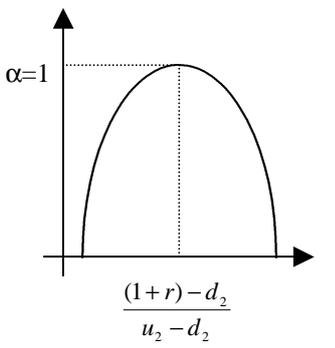
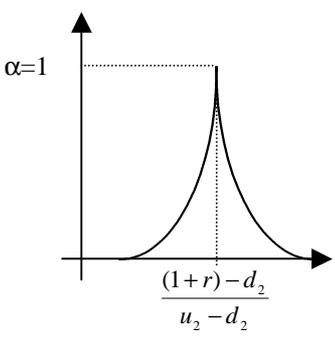
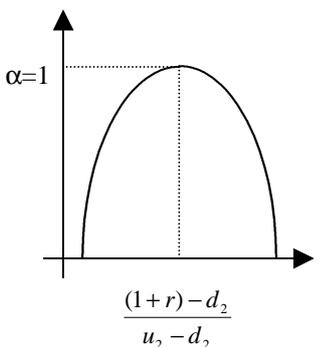
	$u_2 - u_1 > d_2 - d_1$ $u_3 - u_2 < d_3 - d_2$	$u_2 - u_1 < d_2 - d_1$ $u_3 - u_2 > d_3 - d_2$
u, d		
Pd		
Pu		

Table 1. Possible shapes of the artificial probabilities.

The right part:

$$\bar{C}_0 = \frac{P_0 u_3 - X - \alpha P_0 (u_3 - u_2) * \frac{(1+r) - d_1 - \alpha(d_2 - d_1)}{u_1 - d_1 + \alpha(u_2 - u_1 - d_2 + d_1)}}{1+r}$$

is increasing in  $\alpha$  and is concave or convex depending on the sign of the quantity  $(u_2 - d_2)(P_0 u_3 - X) - (u_1 - d_1)(P_0 u_2 - X)$  (for the proofs see Appendix 2). Note that if  $(u_2 - d_2)(P_0 u_3 - X) = (u_1 - d_1)(P_0 u_2 - X)$ , then is linear in  $\alpha$ .

## 5 A multiperiod binomial tree

The aim of this section is to extend the pricing

methodology proposed in the previous section first to a two period and then to a multiperiod binomial model. We will restrict our attention to the case in which the up and down factors are the same at every stage, so that the tree recombines. The up and down jump factors,  $u=(u_1, u_2, u_3)$  and  $d=(d_1, d_2, d_3)$  respectively, are still represented by the two triangular fuzzy numbers identified by the characteristic function in equation (1).

The stock price at  $t=1$  is given by either  $P_0 d$  or  $P_0 u$ . Since  $u$  and  $d$  are triangular fuzzy numbers, it follows that the stock price at  $t=1$  in each state is represented by a triangular fuzzy number, in particular  $P_0 d = (P_0 d_1, P_0 d_2, P_0 d_3)$ , and  $P_0 u = (P_0 u_1, P_0 u_2, P_0 u_3)$ . At time  $t=2$  each of

$P_0d$  and  $P_0u$  may jump again up or down. Applying the rules of multiplication among triangular fuzzy numbers we do not have as a result a triangular fuzzy number [7]. To simplify the algebra, we use the approximate fuzzy numbers, that have the same support and the same peak of the fuzzy number resulting from the multiplication, but they are still triangular:  $P_0d u = P_0u d = (P_0d_1 u_1, P_0d_2 u_2, P_0d_3 u_3)$ ,  $P_0d d = (P_0d_1^2, P_0d_2^2, P_0d_3^2)$ ,  $P_0u u = (P_0u_1^2, P_0u_2^2, P_0u_3^2)$ .

With the same approximation, at time  $t > 2$ , the stock price, will be at each node equal to the following triangular fuzzy number:  $P_0d^i u^j = (P_0d_1^i u_1^j, P_0d_2^i u_2^j, P_0d_3^i u_3^j)$ , with  $i, j = 0, \dots, t$ . We can observe that the

binomial tree is very similar to the standard case in which the jump factors are not fuzzy, the main difference being the stock price that, starting from a crisp level at  $t=0$ , becomes fuzzy at the next stages. We have triangular fuzzy numbers in each state for  $t > 0$ .

## 6 The pricing of an European call option

In this section we show how to price a European call option, with strike price  $X$ , written on the fuzzy stock. Analogously to the one period model, we assume that the strike price satisfies the following condition:

$$P_0d_3^{j+1}u_3^{n-j-1} \leq X \leq P_0d_1^j u_1^{n-j} \quad j = 0, \dots, n-1.$$

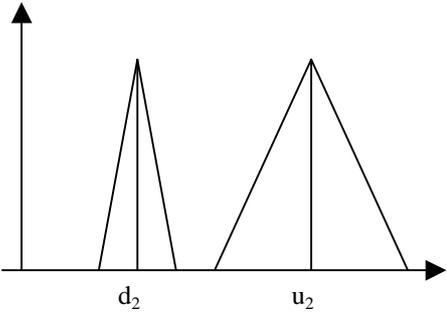
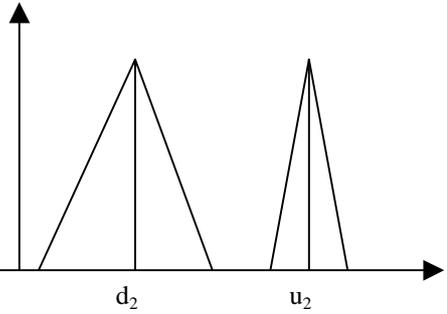
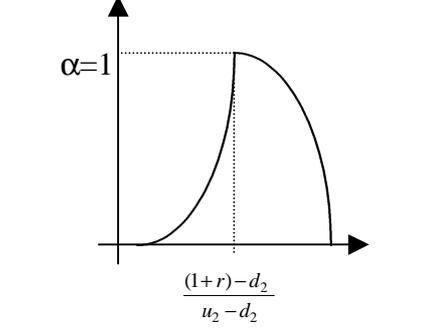
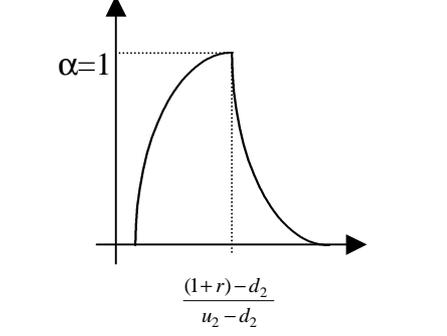
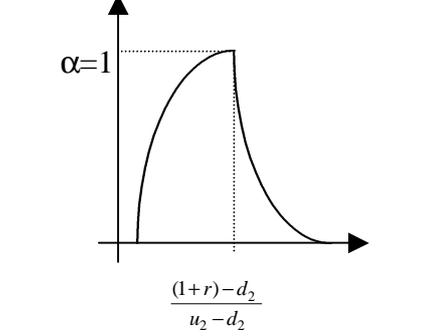
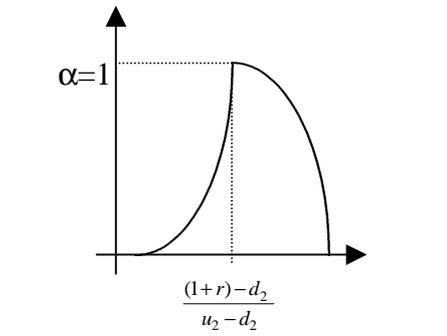
	$u_2 - u_1 > d_2 - d_1$ $u_3 - u_2 > d_3 - d_2$	$u_2 - u_1 < d_2 - d_1$ $u_3 - u_2 < d_3 - d_2$
u, d		
Pd		
Pu		

Table 2. Possible shapes of the artificial probabilities.

For simplicity, we propose the pricing of a call in a two periods model, the extension to n periods is straightforward. We denote the call payoff in state “up-up” with  $C(uu)$ , in state “up-down” and “down-up” with  $C(du)$  in state “down-down” with  $C(dd)$ . Suppose that  $P_0 d_3 u_3 \leq X \leq P_0 u_1^2$ , then the option has a positive payoff only in state up-up, it has zero payoff in the other two states:  $C(dd) = 0$   $C(du) = 0$  and  $C(uu) = (P_0 u u - X)$ . Applying the algebra of fuzzy numbers we obtain the call payoff at time 2, in state “up-up” which is still a triangular fuzzy number equal to:

$$C(uu) = (P_0 u_1^2 - X, P_0 u_2^2 - X, P_0 u_3^2 - X).$$

As in the standard approach, we can price the call by backward induction, starting from the payoff of the call at the expiration date  $t=2$ . To determine the call price at time 1, we note that  $C(d)$  should be equal to zero, since both  $C(dd)$  and  $C(du)$  are equal to zero.  $C(u)$  is given by means of the risk neutral valuation approach, as follows:

$$C(u) = \frac{1}{1+r} \hat{E}[C_2] = \frac{1}{1+r} [p_d * C(ud) + p_u * C(uu)]$$

where  $\hat{E}$  stands for expectation under the risk-neutral probabilities and  $C_2$  is the payoff of the call at  $t=2$ . Since the call has zero payoff in the “up-down” state, the option price formula simplifies to:

$$C(u) = \frac{1}{1+r} [p_u * C(uu)]$$

$$C(u) = [\underline{C}(u), \bar{C}(u)] = \quad (15)$$

$$\left[ \frac{P_0 u_1^2 - X + \mathbf{a} P_0 (u_2^2 - u_1^2)}{1+r} * \frac{(1+r) - d_3 + \mathbf{a}(d_3 - d_2)}{u_3 - d_3 - \mathbf{a}(u_3 - u_2 - d_3 + d_2)}, \right.$$

$$\left. \frac{P_0 u_3^2 - X - \mathbf{a} P_0 (u_3^2 - u_2^2)}{1+r} * \frac{(1+r) - d_1 - \mathbf{a}(d_2 - d_1)}{u_1 - d_1 + \mathbf{a}(u_2 - u_1 - d_2 + d_1)} \right]$$

We now determine the call price  $C_0$  as follows:

$$C_0 = \frac{1}{1+r} \hat{E}[C_1] = \frac{1}{1+r} [p_d * C(d) + p_u * C(u)]$$

where  $\hat{E}$  stands for expectation under the risk-neutral probabilities and  $C_1$  is the payoff of the call at  $t=1$ . Since the call has zero payoff in the down state, the option price formula simplifies to:

$$C_0 = \frac{1}{(1+r)^2} [p_u^2 * C(uu)]$$

$$C_0 = [\underline{C}_0, \bar{C}_0] = \frac{1}{(1+r)^2} *$$

$$\left[ (P_0 u_1^2 - X + \mathbf{a} P_0 (u_2^2 - u_1^2)) * \left( \frac{(1+r) - d_3 + \mathbf{a}(d_3 - d_2)}{u_3 - d_3 - \mathbf{a}(u_3 - u_2 - d_3 + d_2)} \right)^2, \right.$$

$$\left. (P_0 u_3^2 - X - \mathbf{a} P_0 (u_3^2 - u_2^2)) * \left( \frac{(1+r) - d_1 - \mathbf{a}(d_2 - d_1)}{u_1 - d_1 + \mathbf{a}(u_2 - u_1 - d_2 + d_1)} \right)^2 \right]$$

If  $\alpha=0$  the price interval is the largest:

$$C_0 = \frac{1}{(1+r)^2} \left[ (P_0 u_1^2 - X) * \left( \frac{(1+r) - d_3}{u_3 - d_3} \right)^2, (P_0 u_3^2 - X) * \left( \frac{(1+r) - d_1}{u_1 - d_1} \right)^2 \right].$$

If  $\alpha=1$  the interval collapses into only one value:

$$C_0 = \frac{P_0 u_2^2 - X}{(1+r)^2} * \left( \frac{(1+r) - d_2}{u_2 - d_2} \right)^2.$$

As expected, this is the same result of a standard two period binomial option pricing model with  $u_2$  and  $d_2$  as crisp up and down jump factors respectively.

## 7 The defuzzification procedure

Once we have computed the call option price it may be convenient for operative purposes to find a crisp number that synthesises the call option weighted interval. This type of problem is known in the literature as “defuzzification procedure”. There are many methods that, depending on the kind of fuzzy number that we want to defuzzify, provide a scalar that best represents the information contained [3], [7]. In this paper we use a method that is based on the intuitive idea that the best defuzzifier is the scalar that is “closest” to the fuzzy number. We choose this method in particular, for its simplicity and intuitiveness. High on the research agenda is the development of alternative methods for defuzzification.

Define a metric  $D$  as the distance between a fuzzy number  $C$  and a crisp number  $x_0$ , as follows:

$$D(C, x) = \int_0^1 (\underline{C} - x)^2 da + \int_0^1 (\bar{C} - x)^2 da$$

where  $\underline{C}$  and  $\bar{C}$  are the left and right part of the Call price respectively.

Since we look for the scalar  $x$  that minimises the distance with the Call price, we solve the following problem:

$$\min_x D(C, x) = \int_0^1 (\underline{C} - x)^2 da + \int_0^1 (\bar{C} - x)^2 da$$

From the first order condition:

$$\frac{\partial D}{\partial x_0} D(C, x_0) = 0 \text{ we get: } x_0 = \frac{1}{2} \int_0^1 (\underline{C} + \bar{C}) da$$

$x_0$  is the scalar that is closer to the left and right part of the Call price.

For example, if the Call price is a triangular fuzzy number  $C=(c_1, c_2, c_3)$ , we easily get

$$x_0 = \frac{1}{4} (c_1 + 2c_2 + c_3).$$

Once the value of the scalar  $x_0$ , is determined, we compute the numerical value of the distance  $D$ :

$$D(C, x_0) = \int_0^1 (\underline{C} - x_0)^2 da + \int_0^1 (\bar{C} - x_0)^2 da$$

that after some algebra is:

$$D(C, x_0) = \int_0^1 (\underline{C}^2 + \overline{C}^2) d\mathbf{a} - 2x_0^2$$

thus, we actually have an index of the dispersion of the call price around the crisp defuzzifier, that may be interpreted as an index of fuzziness present in the model.

Using the standard binomial option pricing model, we would have obtained a price equal to the peak of the fuzzy call option price, (since for  $\alpha=1$  our model collapses to the standard binomial model). It follows that when the defuzzifier is equal to the peak value of the call price the standard binomial model and our model lead to the same result. This happens for example if the call price is represented by a symmetric triangular fuzzy number. In this case the amount of evidence that the call price is less than the peak value is equal to the amount of evidence that the call price is more than it. As a consequence the peak value is the best representative of the distribution.

The defuzzifier is in general different from the standard binomial price and it is a more reliable price because it takes into account all the information present in the market, regarding the possible amount of increase or decrease of the stock price. Taking into account only a crisp estimate of the parameters, as in a standard binomial model, may result in a loss of information and in a wrong estimate of the call price. This is the reason why this model can help traders in detecting arbitrage opportunities in the real markets.

## 8 Conclusions

In this paper we have proposed a new framework for option pricing, in a multi-period binomial model which is appropriate when there is uncertainty on the states of the world. Our approach hinges on a characterisation of imprecision by means of possibility distributions. Specifically, we assume that the underlying stock price at each node of the binomial tree is opaque to the investors and is modelled by the use of triangular fuzzy numbers. Real options, for the imprecise nature of their underlying might be a valid setting for using this model.

The pricing methodology is still based on a risk neutral valuation approach, whereby weighted intervals of risk-neutral probabilities are used. These intervals of probabilities arise because of the uncertainty on the magnitude of the two possible states up and down of the binomial tree, even if the real probabilities of the stock price jumps are crisp and known in advance. The possibility distribution given on each of the two possible jump of the asset induces in turn a possibility distribution on each of the risk-neutral probabilities. As a consequence, we get a weighted expected value interval for the call price. With a defuzzification procedure, we are able to determine both a scalar that best represents

the distribution of the call price and an index of the fuzziness present in the model. Our methodology offers some advantages. First, it provides an intuitive way to look at the uncertainty in the stock price jumps. Second, it includes the results of the Standard Binomial Option Pricing Model. Third, it traces back the need of using intervals of risk-neutral probabilities, to the opacity in the two possible jumps of the stock. Finally, using weighted intervals of probabilities, i.e. possibility distributions on the risk-neutral probabilities, it provides a weighted expected value interval for the call price and thus we are able to determine a “most likely” value of the call within the interval. High on the research agenda are the development of alternative defuzzification procedures, the implementation of the model with market data, and the extension to a continuous time model.

## Appendix 1 Properties of the risk-neutral probabilities.

Recall that the following inequalities hold:

$$d_1 < d_2 < d_3 < (1+r) < u_1 < u_2 < u_3 \quad (\text{A3.1})$$

$$0 < \underline{p}_d < 1, \quad 0 < \overline{p}_d < 1,$$

$$0 < \underline{p}_u < 1, \quad 0 < \overline{p}_u < 1 \quad (\text{A3.2})$$

Analysing the behaviour of:

$$\underline{p}_u = \frac{(1+r) - d_3 + \mathbf{a}(d_3 - d_2)}{u_3 - d_3 - \mathbf{a}(u_3 - u_2 - d_3 + d_2)} = \frac{N}{D}$$

The first derivative with respect to  $\alpha$  is:

$$\underline{p}'_u = \frac{(d_3 - d_2)(D - N) + N(u_3 - u_2)}{D^2}$$

Given (A3.1) and (A3.2) the sign is always positive.

$$\text{The second derivative is: } \underline{p}''_u = \frac{+ \underline{p}'_u 2(u_3 - u_2 - d_3 + d_2)}{D}$$

the sign clearly depends on the quantity  $(u_3 - u_2 - d_3 + d_2)$ ,

in particular, if  $u_3 - u_2 > d_3 - d_2$  the second derivative is positive; in the opposite case it is negative.

Analysing the behaviour of:

$$\overline{p}_u = \frac{(1+r) - d_1 - \mathbf{a}(d_2 - d_1)}{u_1 - d_1 + \mathbf{a}(u_2 - u_1 - d_2 + d_1)} = \frac{P}{Q}$$

The first derivative with respect to  $\alpha$  is:

$$\overline{p}'_u = \frac{-(d_2 - d_1)(Q - P) - P(u_2 - u_1)}{Q^2}$$

Given (A3.1) and (A3.2) the sign is always negative.

$$\text{The second derivative is: } \overline{p}''_u = \frac{- \overline{p}'_u 2(u_2 - u_1 - d_2 + d_1)}{Q}$$

the sign clearly depends on the quantity  $(u_2 - u_1 - d_2 + d_1)$ ,

in particular, if  $u_2 - u_1 > d_2 - d_1$  the second derivative is positive; in the opposite case it is negative.

Analysing the behaviour of:

$$\underline{p}_d = \frac{u_1 + \mathbf{a}(u_2 - u_1) - (1+r)}{u_1 - d_1 + \mathbf{a}(u_2 - u_1 - d_2 + d_1)} = \frac{R}{S}$$

The first derivative with respect to  $\alpha$  is clearly positive, given (A3.1) and (A3.2):

$$\underline{p}'_d = \frac{(u_2 - u_1)(S - R) + R(d_2 - d_1)}{S^2}$$

$$\text{The second derivative is: } \underline{p}''_d = \frac{-\underline{p}'_d 2(u_2 - u_1 - d_2 + d_1)}{S}$$

the sign clearly depends on the quantity  $(u_2 - u_1 - d_2 + d_1)$ , in particular, if  $u_2 - u_1 > d_2 - d_1$  the second derivative is negative; in the opposite case it is positive.

Analysing the behaviour of:

$$\bar{p}_d = \frac{u_3 - \mathbf{a}(u_3 - u_2) - (1 + r)}{u_3 - d_3 - \mathbf{a}(u_3 - u_2 - d_3 + d_2)} = \frac{T}{Z}$$

The first derivative with respect to  $\alpha$  is:

$$\bar{p}'_d = \frac{-(u_3 - u_2)(Z - T) - (d_3 - d_2)T}{Z^2}$$

and is clearly negative given (A3.1) and (A3.2).

$$\text{The second derivative is: } \bar{p}''_d = \frac{+\bar{p}'_d 2(u_3 - u_2 - d_3 + d_2)}{Z}$$

the sign clearly depends on the quantity  $(u_3 - u_2 - d_3 + d_2)$ , in particular, if  $u_3 - u_2 > d_3 - d_2$  the second derivative is negative; in the opposite case it is positive.

## Appendix 2. Properties of the call price.

Recall that the following inequalities hold:

$$d_1 < d_2 < d_3 < (1+r) < u_1 < u_2 < u_3 \quad (\text{A4.1})$$

$$0 < \underline{p}_d < 1, 0 < \bar{p}_d < 1, 0 < \underline{p}_u < 1, 0 < \bar{p}_u < 1 \quad (\text{A4.2})$$

$$P_0 d_3 \leq X \leq P_0 u_1 \quad (\text{A4.3})$$

Analysing the left bound of the call price in equation (9) the first derivative with respect to  $\alpha$  is:

$$\underline{C}'_0 = \frac{P_0(u_2 - u_1)}{1+r} * \underline{p}'_u + \underline{p}'_u \frac{P_0 u_1 - X + \mathbf{a} P_0(u_2 - u_1)}{1+r} > 0$$

and is positive given A4.1, A4.2 and A4.3 and since  $\underline{p}'_u$  is positive (see appendix 1). The second derivative is:

$$\underline{C}''_0 = \frac{2\underline{p}''_u}{(1+r)[u_3 - d_3 - \mathbf{a}(u_3 - u_2 - d_3 + d_2)]} *$$

$$[(u_3 - d_3)(P_0 u_2 - X) - (u_2 - d_2)(P_0 u_1 - X)]$$

Since the fraction is always positive, the sign depends on the quantity in brackets: if  $[(u_3 - d_3)(P_0 u_2 - X) - (u_2 - d_2)(P_0 u_1 - X)] > 0$  then the sign is positive, in the opposite case it is negative. Note that if  $(u_3 - d_3)(P_0 u_2 - X) = (u_2 - d_2)(P_0 u_1 - X)$  then  $\underline{C}_0$  is linear in  $\alpha$ .

Analysing the shape of the right part we have that the first derivative with respect to  $\alpha$  is:

$$\bar{C}'_0 = -\frac{P_0(u_3 - u_2)}{1+r} \bar{p}'_u + \bar{p}'_u \frac{P_0 u_3 - X - \mathbf{a} P_0(u_3 - u_2)}{1+r} < 0$$

and is negative given A4.1, A4.2 and A4.3 and since  $\bar{p}'_u$  is negative (see Appendix 1). The second derivative is:

$$\bar{C}''_0 = -\frac{2\bar{p}''_u}{(1+r)[u_1 - d_1 + \mathbf{a}(u_2 - u_1 - d_2 + d_1)]} *$$

$$[(u_2 - d_2)(P_0 u_3 - X) - (u_1 - d_1)(P_0 u_2 - X)]$$

Since the fraction is always positive, the sign depends on the quantity in brackets: if  $[(u_2 - d_2)(P_0 u_3 - X) - (u_1 - d_1)(P_0 u_2 - X)] > 0$  then the sign is positive, in the opposite case it is negative. Note that if  $(u_2 - d_2)(P_0 u_3 - X) = (u_1 - d_1)(P_0 u_2 - X)$  then  $\bar{C}_0$  is linear in  $\alpha$ .

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