Random correspondences as bundles of random variables

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Abstract

We prove results that relate random correspondences with their measurable selections, thus providing a foundation for viewing random correspondences as "bundles" of random variables.

Keywords. Random correspondences, random sets, Choquet capacities.

1 Introduction

Since the seminal works of Dempster (1967, 1968), Kendall (1974), and Matheron (1975), random correspondences have been widely used as a generalization of standard random variables. Given two measurable spaces (S, Σ) and $(X, \mathcal{B})$, while random variables associate to elements of S single elements of X, random correspondences relax this assumption by associating nonempty subsets of X to elements of S. This added flexibility turned out to be useful in several areas and we refer the interested reader to the original works, as well as to the recent surveys in Stoyan, Kendall, and Mecke (1995) and Barndorff-Nielsen, Kendall, and van Lieshout (1999). For example, building on Dempster's works, random correspondences have been recently used in Bayesian decision theory to model unforeseen contingencies (see Mukerji (1997) and Ghirardato (2000)).

Given a probability $P : \Sigma \to [0,1]$, a suitably measurable random correspondence $F : S \to 2^X$ induces a lower distribution ν and upper distribution $\overline{\nu}$ on X as follows:

$$\begin{aligned} \nu(A) &= P(\{s : F(s) \subseteq A\}) \text{ and} \\ \overline{\nu}(A) &= P(\{s : F(s) \cap A \neq \varnothing\}) \end{aligned}$$

for all subsets $A \in \mathcal{B}$. In the special case of a random variable $f : S \to X$, we have $\nu(A) = \overline{\nu}(A)$ for all $A \in \mathcal{B}$ and ν reduces to the standard probability distribution P_f induced by f.

The distributions ν and $\overline{\nu}$ therefore generalize the usual probability distributions induced by random variables. The purpose of our work is to study these distributions and in particular their relationships with the standard probability distributions induced by the measurable selections of the random correspondence F.

Specifically, let $S(F)$ be the set of all measurable selections of the random correspondence F, that is, the set of all random variables $h : S \to X$ such that $h(s) \in F(s)$ for all $s \in S$. Each selection $h \in S(F)$ induces a probability distribution P_h on X defined by $P_h(A) = P(\{s : h(s) \in A\})$ for all $A \in \mathcal{B}$. Our purpose is to relate the distributions ν and $\overline{\nu}$ with the set $\{P_h : h \in S(F)\}$ of the standard probability distributions induced by the selections of F. In this we follow Aumann (1965)'s lead, who showed that a fruitful way to look at correspondences is as "bundles" of their selections, a standpoint that makes it possible to relate correspondences with the more familiar single-valued functions. In a probabilistic setting, we adopt a similar view and we provide a connection between the generalized distributions ν and $\overline{\nu}$ and the standard probability distributions $\{P_h : h \in S(F)\}$ that are naturally associated with a random correspondence F.

We have two main results. Consider a real-valued function $u : X \to \mathbb{R}$ defined on the space X, that in applications will be in general the space of interest – e.g., a space of consequences. Since ν and $\overline{\nu}$ are non-additive set functions, we have to consider their Choquet integrals $\int u d\nu$ and $\int u d\overline{\nu}$, that we define in the next section. Our first result, Theorem 1, shows that $\int u d\nu$ and $\int u d\overline{\nu}$ are, respectively, the lower and upper envelopes of the sets of the standard integrals $\left\{\int u dP_h : h \in S(F)\right\}$. That is, we prove that

$$\begin{aligned} \int u d\nu &= \inf\left\{\int u dP_h : h \in S(F)\right\}, \\ \int u d\overline{\nu} &= \sup\left\{\int u dP_h : h \in S(F)\right\}, \end{aligned}$$

provided X is Polish and F compact-valued, conditions often satisfied in applications.

Our second main result, Corollary 1, considers *core* (ν), the set of all countably additive probability measures that setwise dominate ν. This set is often associated with the distribution ν, and Corollary 1 shows that it is nothing but the weak*-closed convex hull of the set $\{P_h : h \in S(F)\}$ of induced probability distributions. That is,

$$core\,(\nu) = \overline{co}^{w^*}\left(\{P_h : h \in S(F)\}\right).$$

These two results (as well as Corollary 2) show that there exists a tight connection between the generalized distributions ν and $\overline{\nu}$ and the standard probability distributions $\{P_h : h \in S(F)\}$ that are naturally associated with them. In this way, we can relate these generalized notions with more familiar standard notions and offer a novel perspective on random correspondences as "bundles" of random variables.

We close by mentioning that in our derivation we obtain two results of some independent interest: a change of variable formula for the Aumann integral and a lemma that generalizes the classic Lusin Theorem to Choquet capacities.

2 Preliminaries

Let Σ be an event σ-algebra of a state space S, and X a metric space with Borel σ-algebra $\mathcal{B}$. As usual in Probability Theory, we will often assume that X is a Polish space, i.e., a separable and complete metric space.

We denote by $ca\,(\mathcal{B})$ the set of all countably additive measures on $\mathcal{B}$ that are bounded with respect to the variation norm. Probabilities are the positive and normalized elements of $ca\,(\mathcal{B})$ that take on value 1 on X. On $ca\,(\mathcal{B})$ we use the weak*-topology induced by $C_b\,(X)$, the set of all continuous and bounded functions $f : X \to \mathbb{R}$. In particular, a net of probabilities $\{p_\alpha\}_\alpha \subseteq ca\,(\mathcal{B})$ weak*-converges to a $p \in ca\,(\mathcal{B})$ if $\lim_\alpha \int f dp_\alpha = \int f dp$ for all $f \in C_b\,(X)$.

A capacity $\nu : \mathcal{B} \to [0, 1]$ is a set function such that:

1. $\nu\,(\varnothing) = 0$ and $\nu\,(X) = 1$;
2. $\nu\,(A) \leq \nu\,(B)$ for all $A, B \in \mathcal{B}$ such that $A \subseteq B$.

The capacity ν is *convex* if $\nu\,(A \cup B) + \nu\,(A \cap B) \geq \nu\,(A) + \nu\,(B)$ for all $A, B \in \mathcal{B}$. We denote by *core* (ν) the core of a capacity ν, i.e., the set $\{p \in ca\,(\mathcal{B}) : p\,(X) = 1 \text{ and } p\,(A) \geq \nu\,(A)\, \forall A \in \mathcal{B}\}$ of all probabilities that setwise dominate the capacity.

The notion of integral associated with capacities is the Choquet integral. Given a measurable real-valued function $u : X \to \mathbb{R}$, the Choquet integral $\int u d\nu$ is defined by

$$\int u d\nu = \int_0^{+\infty} \nu\,(u \geq t)\, dt + \int_{-\infty}^0 [\nu\,(u \geq t) - 1]\, dt,$$

where the right hand side is a Riemann integral, which is well defined since $\nu\,(u \geq t)$ is a monotone function in t.

A correspondence $F : S \to 2^X$ associates nonempty subsets of X to states of S. The strong inverse F^{-1} of F is defined by $F^{-1}(A) = \{s : F(s) \subseteq A\}$ for all sets $A \subseteq X$, while the weak inverse F^w is defined by $F^w(A) = \{s : F(s) \cap A \neq \varnothing\}$. Since $F^w(A) = S - F^{-1}(A^c)$ for all $A \subseteq X$, in general it will be enough to focus on F^{-1}.

Using a standard notion of measurability, we now introduce random correspondences.

Definition 1 *A correspondence $F : S \to 2^X$ is $\mathcal{B}$-measurable if $F^{-1}(A) \in \Sigma$ for all Borel sets $A \in \mathcal{B}$. A correspondence $F : S \to 2^X$ which is $\mathcal{B}$-measurable is called a random correspondence.*

If the values of F are singletons, random correspondences reduce to standard random variables. Random correspondences are closely related to random sets, the only difference being in the notion of measurability used. The main reason why we use random correspondences is to have the distribution ν defined on the entire σ-algebra $\mathcal{B}$. In any event, in Section 4 we show that our main result holds for random sets as well.

Given a probability measure $P : \Sigma \to [0, 1]$ on the state space, a random variable $f : S \to X$ induces a distribution $P_f : \mathcal{B} \to [0, 1]$ defined by $P_f(A) = P(\{s : f(s) \in A\})$ for all $A \in \mathcal{B}$. In a similar way, random correspondences induce a lower distribution $\nu : \mathcal{B} \to [0, 1]$ and an upper distribution $\overline{\nu} : \mathcal{B} \to [0, 1]$ defined by:

$$\begin{aligned}
\nu\,(A) &= P\left(F^{-1}(A)\right) = P\left(\{s : F(s) \subseteq A\}\right) \\
\overline{\nu}\,(A) &= P\left(F^w(A)\right) = P\left(\{s : F(s) \cap A \neq \varnothing\}\right)
\end{aligned}$$

for all $A \in \mathcal{B}$. Since $\overline{\nu}\,(A) = 1 - \nu\,(A^c)$ for all $A \in \mathcal{B}$, there is a simple duality between the two distributions, and in the sequel we will mostly focus on ν.

Unlike the distributions P_f, the set function ν is in general non-additive. However, it is totally monotone, i.e.,

$$\nu\left(\bigcup_{i=1}^n A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1}\, \nu\left(\bigcap_{i \in I} A_i\right)$$

for all $\{A_i\}_{i\in\{1,\ldots,n\}} \subseteq \mathcal{B}$ and for all $n \in \mathbb{N}$. In fact, it is easy to check that the probability distribution induced by F when viewed as an usual function from S to 2^X is the Moebius transform of ν. Besides total monotonicity, the distributions ν have other important properties. Following Kuratowski (1966) we say that a monotone increasing sequence $\{A_n\}_{n\geq 1}$ of Borel sets is strictly monotone if $A_n \subseteq int\,(A_{n+1})$ for all $n \geq 1$ (e.g., all sets A_n are open).

Proposition 1 *Let $\nu : \mathcal{B} \to [0,1]$ be the distribution induced by a random correspondence. Then:*

(i) $\lim_{n\to\infty} \nu(A_n) = \nu\left(\bigcap_{n\geq 1} A_n\right)$ *for all non-increasing sequences of Borel sets.*

If, in addition, the correspondence has compact values, then:

(ii) $\lim_{n\to\infty} \nu(A_n) = \nu\left(\bigcup_{n\geq 1} A_n\right)$ *for all non-decreasing strictly monotone sequences of Borel sets.*

Point (i) is easy to check (see, e.g., Nguyen (1978)). Point (ii) is in general false if the sequence is not strictly monotone, as the following example shows.

Example. Let $X = [0,1]$ and $K = [1/2, 1]$. Consider the multifunction $F : S \to 2^X$ defined by $F(s) = K$ for all $s \in S$. Then ν is $\{0,1\}$-valued and $\nu(A) = 1$ if and only if $K \subseteq A$.[1] Set $A_n = [0, 1-1/n] \cup \{1\}$. The sequence $\{A_n\}_{n\geq 1}$ is not strictly monotone and so Proposition 1 does not apply. We have $A_n \uparrow X$, but $\lim_n \nu(A_n) \neq \nu(X)$. In fact, $\lim_n \nu(A_n) = 0$.

The next result shows some regularity properties of the distribution ν. Parts of this result are more or less known, though we did not find a reference for the result in this generality.[2] For this reason we provide a proof.

Proposition 2 *The distribution ν induced by a compact-valued random correspondence is regular, i.e.,*

$$\begin{aligned}\nu(A) &= \sup\{\nu(C) : C \subseteq A \text{ and } C \text{ closed}\} \\ &= \inf\{\nu(G) : A \subseteq G \text{ and } G \text{ open}\}\end{aligned}$$

for all Borel sets A. If, in addition, X is Polish, then ν is tight, i.e., $\nu(A) = \sup\{\nu(K) : K \subseteq A$ and K compact$\}$ for all Borel sets A.

[1] In other words, ν is the unanimity game u_K.

[2] For example, the Polish space part is an immediate consequence of an unproved observation on p. 253 of Huber and Strassen (1973).

3 Main results

In this section we characterize the random correspondence $F : S \to 2^X$ via the probability distributions induced by its measurable selections. The first result, which is our main result, shows that the Choquet integral relative to ν can be expressed in terms of the standard integrals associated with the probability distributions induced by the measurable selections of F.

Before stating the result we introduce a class of functions.

Definition 2 *A real-valued function $u : X \to \mathbb{R}$ is lower (upper, resp.) Weierstrass if it attains its infimum (supremum, resp.) on all compact sets of X.*

The class of lower Weierstrass functions is broad and it includes:

(i) all lower semicontinuous functions $u : X \to \mathbb{R}$;

(ii) all finite-valued functions $u : X \to \mathbb{R}$.

On the other hand, continuous functions and finite-valued functions are examples of functions that are both lower and upper Weierstrass.

We can now state our main result. Recall that $S(F)$ is the set of measurable selections of F.

Theorem 1 *Let ν be the distribution induced by a compact-valued random correspondence $F : S \to 2^X$. If X is a Polish space, then*

$$\int u d\nu = \inf_{h\in S(F)} \int u dP_h \qquad (1)$$

for all bounded and measurable functions $u : X \to \mathbb{R}$. If, in addition, u is lower Weierstrass, then in (1) we have a min *instead of an* inf*.*

Remark. A dual version of Theorem 1 holds, where ν, inf, and lower Weierstrass are replaced respectively with $\overline{\nu}$, sup, and upper Weierstrass.

From a probabilistic standpoint, the set of induced probability distributions $\{P_h : h \in S(F)\}$ is a very important subset of *core* (ν) since it has a direct connection with the random correspondence F. It would be therefore desirable that $\{P_h : h \in S(F)\}$ were also a mathematically important subset of *core* (ν). In general, $\{P_h : h \in S(F)\}$ is not a convex set and so in general *core* $(\nu) \neq \{P_h : h \in S(F)\}$. For example, let $S = X = [0,1]$ and let $F(s) = \{0, s\}$ for all $s \in S$. It can be checked that $\{P_h : h \in S(F)\}$ is not convex.

However, the next result – based on Theorem 1 – shows that $\{P_h : h \in S(F)\}$ is still an important subset of $core(\nu)$. As a matter of fact, $core(\nu)$ is nothing but the weak*-closed convex hull $\overline{co}^{w^*}(\cdot)$ of $\{P_h : h \in S(F)\}$.

Corollary 1 *Let ν be the distribution induced by a compact-valued random correspondence $F : S \to 2^X$. If X is a Polish space, then*

$$core(\nu) = \overline{co}^{w^*}(\{P_h : h \in S(F)\}),$$

and $ext(core(\nu)) \subseteq \overline{\{P_h : h \in S(F)\}}^{w^*}$, i.e., all extreme points of $core(\nu)$ belong to the weak-closure of $\{P_h : h \in S(F)\}$.*

We close with a simple but useful consequence of Theorem 1, that further shows the importance of the set $\{P_h : h \in S(F)\}$ for ν.

Corollary 2 *Let ν be the distribution induced by a compact-valued random correspondence $F : S \to 2^X$. If X is a Polish space, then:*

(i) for each finite chain $\{A_i\}_{i=1}^n$ of Borel sets there exists $h \in S(F)$ such that $\nu(A_i) = P_h(A_i)$ for each $i = 1, ..., n$.

(ii) for each infinite chain $\{G_i\}_{i\in[0,1]}$ of open sets with

(a) $G_i \supseteq G_j$ if $i \geq j$ and $G_0 = \varnothing$,

(b) $\bigcup_{j<i} G_j = G_i$,

there exists $h \in S(F)$ such that $P_h(G_i) = \nu(G_i)$ for all $i \in [0,1]$.

(iii) for each infinite chain $\{C_i\}_{i\in[0,1]}$ of closed sets with

(a) $C_i \subseteq int(C_j)$ if $i \geq j$,

(b) $\bigcap_{j<i} C_j = C_i$,

there exists $h \in S(F)$ such that $P_h(C_i) = \nu(C_i)$ for all $i \in [0,1]$.

4 Additional results

4.1 A change of variable formula for the Aumann integral

Given a correspondence $G : S \to 2^{\mathbb{R}}$, let $\widetilde{S}(G)$ be the set of all P-a.e. measurable selections, i.e., $h \in \widetilde{S}(G)$ if it is measurable and P-a.e. $h(s) \in G(s)$. The Aumann integral $\int G dP$ is then defined as the set

$$\left\{\int h dP : h \text{ integrable and } h \in \widetilde{S}(G)\right\}.$$

Consider the correspondence $u \circ F : S \to 2^{\mathbb{R}}$, the composition of the function $u : X \to \mathbb{R}$ with the correspondence $F : S \to 2^X$. For all $s \in S$, $(u \circ F)(s) = \{u(x) : x \in F(s)\}$.

Lemma 1 *Let $F : S \to 2^X$ be a compact-valued random correspondence. If X is a Polish space, then*

$$\int (u \circ F) dP = \left\{\int u dP_h : h \in S(F)\right\} \tag{2}$$

for all bounded and measurable functions $u : X \to \mathbb{R}$.

Along with Theorem 1, this lemma delivers a change of variable formula for the Aumann integral. Our result complements Theorem 5 of Hildenbrand (1974), that considers the composition of a correspondence with a function; in contrast, we consider the composition of function with a correspondence.

Theorem 2 *Let ν be the distribution induced by a compact-valued random correspondence $F : S \to 2^X$. If X is a Polish space, then*

$$\int u d\nu = \inf \int (u \circ F) dP \tag{3}$$

for all bounded and measurable functions $u : X \to \mathbb{R}$. If, in addition, u is lower Weierstrass, then in (3) we have a min *instead of an* inf*.*

Using Lemma 1 we can get some further information about the set of induced distributions $\{P_h : h \in S(F)\}$ by using some well-known properties of the Aumann integral. First of all, if u is continuous and bounded or finite-valued, the correspondence $u \circ F$ is closed-valued and integrably bounded provided F is compact-valued. Hence, $\int (u \circ F) dP$ is a compact subset of $\mathbb{R}$ (see, e.g., Proposition 7 p. 73 of Hildenbrand (1974)), and we conclude that the set $\left\{\int u dP_h : h \in S(F)\right\}$ is a compact subset of $\mathbb{R}$ provided F is compact-valued and u is continuous and bounded or finite-valued.

Another interesting property of the Aumann integral is that it is convex when P is non-atomic. Along with Theorem 1 and Lemma 1, this immediately implies the following useful result.

Proposition 3 *Let ν be the distribution induced by a compact-valued random correspondence $F : S \to 2^X$. If X is a Polish space and P is non-atomic, then*

$$\left(\int u d\nu, \int u d\overline{\nu}\right) \subseteq \left\{\int u dP_h : h \in S(F)\right\}$$

$$\subseteq \left[\int u d\nu, \int u d\overline{\nu}\right]$$

for all measurable functions $u : X \to \mathbb{R}$. If, in addition, u is lower Weierstrass, then

$$\left[\int u d\nu, \int u d\overline{\nu}\right) \subseteq \left\{\int u dP_h : h \in S(F)\right\}$$

while, if u is upper Weierstrass, then

$$\left(\int u d\nu, \int u d\overline{\nu}\right] \subseteq \left\{\int u dP_h : h \in S(F)\right\}.$$

4.2 The results for random sets

As mentioned before, the difference between random sets and random correspondences lies in the notion of measurability used.

Definition 3 *A correspondence $F : S \to 2^X$ is $\mathcal{G}$-measurable if $F^{-1}(G) \in \Sigma$ for all open sets $G \subseteq X$. A correspondence $F : S \to 2^X$ which is $\mathcal{G}$-measurable is called a random set.*

Remark. In σ-compact Hausdorff spaces, a closed-valued correspondence is a random set if and only if $\{s : F(s) \cap K \neq \varnothing\} \in \Sigma$ for all compact sets K (cf. Himmelberg (1975) Theorem 3.5). In particular, this is true in separable locally compact Hausdorff spaces.

Clearly, all random correspondences are random sets. Though the converse is in general false, the next result, due to Debreu (1967), provides an important case where it holds (cf. Himmelberg (1975) pp. 57-58). Σ_* denotes the completion of Σ under P, i.e., the collection of all sets of the form $A \cup N$, where $A \in \Sigma$ and N is P-null.

Theorem 3 *Let $F : S \to 2^X$ be a closed-valued random set and suppose X is a Polish space. Then $F^{-1}(A) \in \Sigma_*$ for all $A \in \mathcal{B}$, and so F is a random correspondence provided $\Sigma = \Sigma_*$, i.e., provided (S, Σ, P) is a complete measure space.*

Theorem 3 suggests a simple way to extend our results to random sets even when (S, Σ, P) is not a complete measure space. For, let $F : S \to 2^X$ be a closed-valued random set. Its distribution ν is defined only on open sets and we have $\nu(G) = P\left(F^{-1}(G)\right)$ for all open sets $G \subseteq X$. However, let P_* be the unique extension of P to Σ_* and define a set function $\nu_* : \mathcal{B} \to [0,1]$ as follows:

$$\nu_*(A) = P_*\left(F^{-1}(A)\right)$$

for all $A \in \mathcal{B}$. If X is Polish, the set function ν_* is well defined by Theorem 3 and it coincides with ν on the open sets. Actually, more is true. When the metric space X is separable and F is a closed-valued random set, $F^{-1}(C) \in \Sigma$ for all closed sets $C \subseteq X$ (see Theorem 3.5 of Himmelberg (1975)). Hence, $\nu_*(C) = \nu(C)$ also for all closed sets $C \subseteq X$.

We call ν_* the *extended distribution* of F. It satisfies the same properties as the standard distributions induced by random correspondences.

Proposition 4 *Suppose X is Polish, and let $\nu_* : \mathcal{B} \to [0,1]$ the extended distribution induced by a compact-valued random set. Then:*

(i) $\lim_{n\to\infty} \nu_*(A_n) = \nu_*\left(\bigcap_{n\geq 1} A_n\right)$ *for all non-increasing sequences of Borel sets.*

(ii) $\lim_{n\to\infty} \nu_*(A_n) = \nu_*\left(\bigcup_{n\geq 1} A_n\right)$ *for all non-decreasing strictly monotone sequences of Borel sets.*

(iii) *ν_* is regular and tight.*

We can now extend Theorem 1 to random sets.

Proposition 5 *Let X be a Polish space and let ν_* be the extended distribution induced by a compact-valued random set $F : S \to 2^X$. Then*

$$\int u d\nu_* = \inf_{h\in S(F)} \int u dP_h \tag{4}$$

for all bounded and measurable functions $u : X \to \mathbb{R}$. If, in addition, u is lower Weierstrass, then in (4) we have a min *instead of an* inf.

Remark. Since ν and ν_* coincide on all open sets and on all closed sets, $\int u d\nu_* = \int u d\nu$ for all upper semicontinuous and all lower semicontinuous functions $u : X \to \mathbb{R}$.

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