Uncertainty Aversion With Second-Order Probabilities and Utilities

Robert F. Nau
Fuqua School of Business
Duke University
robert.nau@duke.edu

Abstract

Aversion to uncertainty is commonly attributed to non-additivity of subjective probabilities for ambiguous events, as in the Choquet expected utility model. This paper shows that uncertainty aversion can be parsimoniously explained by a simple model of "partially separable" non-expected utility preferences in which the decision maker satisfies the independence axiom selectively within partitions of the state space whose elements have similar degrees of uncertainty. As such, she may behave like an expected-utility maximizer with additive probabilities for assets in the same uncertainty class, while exhibiting higher degrees of risk aversion toward assets that are more uncertain. An alternative interpretation of the same model is that the decision maker may be uncertain about her credal state (represented by second-order probabilities for her first-order probabilities and utilities), and she may be averse to that uncertainty (represented by a second-order utility function). The Ellsberg and Allais paradoxes are explained by way of illustration.

Keywords. Risk aversion, uncertainty aversion, non-additive probabilities, Choquet expected utility.

1 Introduction

The axiomatization of expected utility by von Neumman-Morgenstern and Savage hinges on the axiom of independence, which requires preferences to be separable across mutually exclusive events and leads to representations by utility functions that are additively separable across states of the world. A strong implication of the independence axiom is that preferences are not permitted to depend on qualitative properties of events but rather only on the sum total of the values attached to their constituent states, which in turn depend only on the probabilities of the states and the consequences to which they lead. If the set of states of nature can be partitioned in two or more ways, the decision maker is not permitted to display uniformly different risk attitudes toward acts that are measurable with respect to different partitions, because the events in one partition are, at bottom, composed of the same states as those in any other. There is considerable empirical evidence that individuals violate this requirement in certain kinds of choice situations. A classic example is provided by Ellsberg's 2-color paradox, in which subjects consistently prefer to bet on unambiguous events rather than ambiguous events, even when they are otherwise equivalent by virtue of symmetry, a phenomenon that has come to be known as *uncertainty aversion*. Other violations of independence are provided by Ellsberg's 3-color paradox and Allais' paradox, in which subjects' preferences between two acts that agree in some events depend on *how* they agree there, as though there were complementarities among events.

A variety of models of non-expected utility have been proposed to accommodate violations of the independence axiom, and most of them do so by positing that the decision maker has non-probabilistic beliefs or that her preferences depend nonlinearly on probabilities. The Choquet expected utility model, in particular, assumes that the decision maker tends to overweight events leading to inferior payoffs by applying non-additive subjective probabilities derived from a Choquet capacity. (Schmeidler 1989, Epstein 1999) The ranking of states according to the payoffs to which they lead thus plays a key role in the representation of uncertainty aversion: the decision maker violates the axioms of expected utility theory only when faced with choices among acts whose payoffs induce different rankings of states. Within sets of acts whose payoffs are comonotonic, the decision maker's preferences have an ordinary expected-utility representation. Another way to view the Choquet model is to note that it implies that the decision maker has indifference curves in state-payoff space that are kinked at the boundaries between comonotonic sets of acts. If the decision maker's status quo wealth happens to fall on such a kink—which is a set of measure zero in state-payoff space—her local preferences (i.e., her preferences for "neighboring" acts) will display uncertainty aversion, otherwise they will not.

Other explanations of uncertainty aversion are possible: one approach would be to drop the axiom of completeness and allow the decision maker to have partially ordered preferences represented by convex sets of probabilities. (Walley 1991) Such a person would *always* act as if she were sitting on a kink in an indifference curve, regardless of her status quo wealth, although her preferences among some pairs of acts would be indeterminate. Another approach would be to relax the independence axiom in a different manner so as to permit the decision maker to display different risk attitudes toward different classes of events—or more generally, to allow preferences among acts to depend on how states are "bundled" with other states—without necessarily ruling out the representation of beliefs by additive probabilities. The latter approach is explored in this paper. We will show that a simple model of "partially separable" preferences can explain both Ellsberg's and Allais paradoxes and provide a representation of local aversion to uncertainty.

The organization of the paper is as follows: section 2 introduces the basic mathematical framework and concepts of risk aversion. Section 3 gives a simple example of a utility function defined on a 4-element state space that explains Ellsberg's 2-color and 3-color paradoxes and Allais' paradox. Section 4 presents a more detailed and general version of the same model and the axioms on which it rests. Section 5 discusses how the model differs from Choquet expected utility, and Section 6 presents some concluding comments.

2 Preliminaries

The modeling framework used throughout this paper will be that of *state-preference theory* (Arrow 1953/1964, Debreu 1959, Hirshleifer 1965), which encompasses both expected-utility and non-expected-utility models of choice under uncertainty. Suppose that there are n mutually exclusive, collectively exhaustive, states of nature and a single divisible commodity (money) in terms of which payoffs are measured. The wealth distribution of an individual can then be represented by an n-vector $\boldsymbol{w}$, whose j^{th} element w_j denotes the quantity of money received in state j, in addition to (unobserved) status quo wealth. If the individual's preferences among wealth distributions satisfy the standard axioms of consumer theory (reflexivity, completeness, transitivity, and continuity) then they are represented by an ordinal utility function $U(\boldsymbol{w})$. If no additional restrictions are placed on preferences, the individual is rational according to the usual standards of consumer theory, but she may have "non-expected utility preferences" in the sense that her valuation of a risky asset may not be decomposable into a product of probabilities for states and utilities for consequences. For example, she may behave as if amounts of money received in different states are substitutes or complements for each other, which is forbidden under expected utility theory.

If state-preferences are additionally assumed to satisfy the independence axiom (Savage's P2), then $U(\boldsymbol{w})$ has an additively separable representation:

$$U(\boldsymbol{w}) = v_1(w_1) + v_2(w_2) + \ldots + v_n(w_n).$$

If preferences are further assumed to be conditionally state-independent (Savage's P3), then $U(\boldsymbol{w})$ has a state-independent expected-utility representation:

$$U(\boldsymbol{w}) = p_1u(w_1) + p_2u(w_2) + \ldots + p_nu(w_n),$$

where p is a unique probability distribution and $u(x)$ is a state-independent utility function that is unique up to positive affine scaling, as in Savage's model. Although the probabilities in the latter representation are unique, it does not yet follow that they are the decision maker's "true" subjective probabilities, because there are many other equivalent representations in which different probabilities are combined with state-dependent utilities. In order for the decision maker to be "probabilistically sophisticated"—i.e., in order for her preferences to determine a unique ordering of events by probability—an additional qualitative probability axiom (Savage's P4 or Machina and Schmeidler's P4*, 1992) is needed, together with an *a priori* definition of consequences whose utility is "constant" across states of nature. (Schervish et al. 1990)

If the decision maker's preferences are sufficiently smooth, her utility function U is differentiable and its gradient is a non-negative vector that can be normalized to yield a probability distribution $\boldsymbol{\pi}$, whose j^{th} element is

$$\pi_j = \frac{U_j(\boldsymbol{w})}{\sum_{h=1}^{n} U_h(\boldsymbol{w})} ,$$

where $U_j(\boldsymbol{w})$ denotes the partial derivative $\partial U/\partial w_j$ evaluated at $\boldsymbol{w}$. (Implicitly $\boldsymbol{\pi}$ is a function of $\boldsymbol{w}$, but its wealth argument will be suppressed for notational convenience.) $\boldsymbol{\pi}$ is invariant to monotonic transformations of U and is observable, regardless of whether the decision maker is probabilistically sophisticated. It is commonly known as a *risk neutral probability distribution* because the decision maker prices very small assets in a seemingly risk-neutral manner with respect to it. For a decision maker who is a state-independent expected-utility maximizer, risk neutral probabilities are proportional to the product of true subjective probabilities and relative marginal utilities for money at the current wealth position, i.e.,

$$\pi_j \propto p_j u'(w_j).$$

If an attempt is made to elicit the decision maker's subjective probabilities by de Finetti's method—i.e., by asking which gambles she is willing to accept—the probabilities that are observed will be her risk neutral probabilities rather than her true probabilities. The two distributions will differ if the decision maker has sufficiently large prior stakes in the outcomes of events to affect her marginal utilities for money.

In order to characterize aversion to uncertainty, it is necessary to begin with a characterization of aversion to risk. Following Yaari (1969) the decision maker is defined to be *risk averse* if her state-preferences are convex, which means—as in consumer theory—that her ordinal utility function U must be quasi-concave. Thus, the definition of risk aversion does not require prior definitions of expected value or absence of risk. Following Nau (2001), the decision maker's local risk aversion will be measured by the difference between her risk-neutral valuation of an asset and the price at which she is willing to buy it. Let $\boldsymbol{z}$ denote the payoff vector of a risky asset. The decision maker's *buying price* for $\boldsymbol{z}$, denoted $P_b(\boldsymbol{z})$, is determined by

$$U(\boldsymbol{w}+\boldsymbol{z}-P_b(\boldsymbol{z})) - U(\boldsymbol{w}) = 0.$$

The *buying risk premium* associated with $\boldsymbol{z}$ at wealth $\boldsymbol{w}$, here denoted $\rho_b(\boldsymbol{z})$, is the difference between the asset's risk neutral expected value and its buying price:

$$\rho_b(\boldsymbol{z}) = E_\pi[\boldsymbol{z}] - P_b(\boldsymbol{z}).$$

It follows as a theorem that the decision maker is risk averse if and only if her buying risk premium is non-negative for every asset $\boldsymbol{z}$ at every wealth distribution $\boldsymbol{w}$. The buying risk premium has the convenient property that $\rho_b(\boldsymbol{z}+c)=\rho_b(\boldsymbol{z})$ for any constant c.

The Pratt-Arrow measure of local risk aversion can be readily generalized to the present context if the ordinal utility function U is assumed to be twice differentiable. Let $U_{jk}(\boldsymbol{w})$ denote the second partial derivative $\partial^2 U/\partial w_j w_k$, evaluated at $\boldsymbol{w}$, and define the local *risk aversion matrix* as the matrix $\boldsymbol{R}$ whose jk^{th} element is the negative of the ratio of second to first derivatives:

$$r_{jk} = -U_{jk}(\boldsymbol{w})/U_j(\boldsymbol{w}).$$

(Like $\boldsymbol{\pi}$ and $\rho_b(\boldsymbol{z})$, $\boldsymbol{R}$ implicitly depends on $\boldsymbol{w}$, but its wealth argument will be suppressed for notational convenience.) If $\boldsymbol{z}$ is a neutral asset ($E_\pi[\boldsymbol{z}] = 0$), its risk premium has the following second-order approximation which generalizes the Pratt-Arrow formula:

$$\rho_b(\boldsymbol{z}) \approx \tfrac{1}{2}\, \boldsymbol{z}^{\mathrm{T}} \Pi \boldsymbol{R}\, \boldsymbol{z},$$

where $\Pi = \mathrm{diag}(\boldsymbol{\pi})$. (Nau 2001) In the special case where U has a state-independent expected-utility representation, $\boldsymbol{R}$ is a diagonal matrix and the risk premium formula reduces to

$$\rho_b(\boldsymbol{z}) \approx \tfrac{1}{2} E_\pi[\boldsymbol{r}\, \boldsymbol{z}^2],$$

where $\boldsymbol{r}$ is a vector-valued risk aversion measure whose j^{th} element is

$$r_j = -U_{jj}(\boldsymbol{w})/U_j(\boldsymbol{w}) = -u''(w_j)/u'(w_j).$$

3 A simple model of smooth preferences explaining the Ellsberg and Allais paradoxes

In the 2-color Ellsberg paradox, a subject is presented with two urns, one that is known to contain exactly 50 red and 50 black balls and another that contains red and black balls in unknown proportions, and for a given color she is asked whether she would rather receive a fixed prize (say, $100) conditional on drawing a ball of that color from the "known" urn or from the "unknown" urn. The familiar pattern of results is that, regardless of which color is specified, most subjects prefer to receive the prize conditional on a ball drawn from the known urn. This pattern is inconsistent with any possible assignment of probabilities to events and utilities to prizes, and the subject is now "in trouble with the Savage axioms." The usual interpretation is that subjects are averse to the ambiguity or lack of information associated with the unknown urn and therefore behave as though their probabilities were not additive, providing the motivation for the Choquet expected utility model in which the probability measure is replaced by a non-additive capacity This section introduces a simple preference model that explains both Ellsberg's and Allais's paradoxes in terms of uncertainty aversion "in the small" even with additive probabilities.

Let a_1 [a_2] denote the event that the ball drawn from the *unknown* urn is red [black], and let b_1 [b_2] denote the event that the ball drawn from the *known* urn is red [black]. The relevant state space is then $\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$. Unless the subject has a strict color preference and/or prior stakes in the outcomes of the events (which we assume she does not), the four states are completely symmetric when considered one-at-a-time: each is the conjunction of an ambiguous event (a_1 or a_2) and an unambiguous event (b_1 or b_2) differing only in their color associations. If the subject had to choose *one* of the four states on which to stake a prize, she would have no basis for a strict preference. The paradox lies in the fact that the states are *not* symmetric when considered two-at-a-time: the pair of states$\{a_1b_1, a_2b_1\}$ has an objectively known probability while the pair of states $\{a_1b_1, a_1b_2\}$ does not.

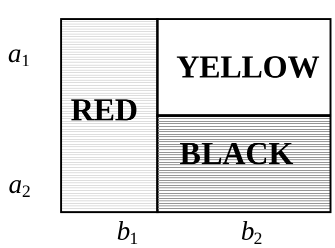
Let $\boldsymbol{w} = (w_{11}, w_{12}, w_{21}, w_{22})$ denote a hypothetical wealth distribution, where w_{ij} is wealth in state a_ib_j, and suppose that the subject evaluates wealth distributions according to the following non-separable utility function:

$$(1)\qquad U(\boldsymbol{w}) = -p_1 \exp(-\alpha(q_{11}w_{11} + q_{12}w_{12})) - p_2 \exp(-\alpha(q_{21}w_{21} + q_{22}w_{22}))$$

where α is a positive constant and $p_1 = p_2 = q_{11} = q_{12} = q_{21} = q_{22} = \tfrac{1}{2}$. It is natural to interpret p_i as a marginal probability for a_i and q_{ij} as a conditional probability of b_j given a_i, with independence between A and B. Suppose that the prior wealth distribution is an arbitrary constant—i.e., the subject has no prior stakes in the draws from either urn. Then the states are symmetric with respect to changes in wealth one-state-at-a-time, from which it follows that $\boldsymbol{\pi} = (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4})$. Hence, for infinitely small bets, the subject does not distinguish among the states, exactly as if she were a state-independent expected utility maximizer with uniform prior probabilities and no prior stakes. However, she *does* distinguish among the states when considering bets in which states are grouped together and in which the stakes are large enough for risk aversion to come into play. For example, the subject would prefer to pair a finite gain in state a_1b_1 with an equal loss in state a_1b_2 (yielding no change in U) rather than with an equal loss in state a_2b_1 or state a_2b_2 (yielding a decrease in U). The risk aversion matrix in this case is $\boldsymbol{R} = ½\,\alpha\,\boldsymbol{C}$, where $\boldsymbol{C}$ has a block structure with 1's in its upper left and lower right 2×2 submatrices.

Now consider the following three neutral bets. In bet #1, the subject wins x>0 if a red ball is drawn from the *known* urn and loses x otherwise, so that the payoff vector is (x, -x, x, -x). In bet #2, the subject wins x if a red ball is drawn from the *unknown* urn and loses x otherwise, so that the payoff vector is (x, x, -x, -x). In bet #3, the subject wins x if the balls drawn from urns 1 and 2 are the *same color*, and loses x otherwise, so that the payoff vector is (x, -x, -x, x). Applying the formula $\rho_b(\boldsymbol{z}) = ½\,\boldsymbol{z}^{\mathrm{T}}\Pi\boldsymbol{R}\boldsymbol{z}$, the risk premium for bet #1 is zero, and the same is true if the payoffs are reversed so that black is the winning color. Hence, *the subject is risk neutral with respect to bets on the known urn*. Whereas, the risk premium for bet #2 is $½\alpha x^2$, and the same risk premium is obtained if the winning color is changed to black. Hence, *the subject is risk averse with respect to bets on the unknown urn*: she behaves toward it as if she believes red and black are equally likely but her Pratt-Arrow risk aversion coefficient is equal to α. This pattern of risk neutral behavior toward unambiguous events and risk averse behavior toward ambiguous events is clearly inconsistent with subjective expected utility theory, but it does not expose the subject to arbitrage because U is an increasing function of wealth in every state. Interestingly, the risk premium for bet #3 is the same as for bet #1, namely zero, and the same is true if the payoffs are reversed so that the subject wins if the balls are of different colors. Hence, the subject behaves risk neutrally with respect to the unknown urn when the winning color is determined by "objective" randomization using the known urn, which is a well-known trick for eliminating the ambiguity.

In the 3-color Ellsberg problem, the subject chooses among bets on the color of a ball drawn from a single urn containing 30 red balls and a total of 60 black and yellow balls in unknown proportions. The uncertainty about the color of the ball can be modeled by a 2×2 partition of states with the color mapping $b_1 \rightarrow$ Red, $a_1 \cap b_2 \rightarrow$ Yellow, and $a_2 \cap b_2 \rightarrow$ Black, as shown in the following schematic diagram:



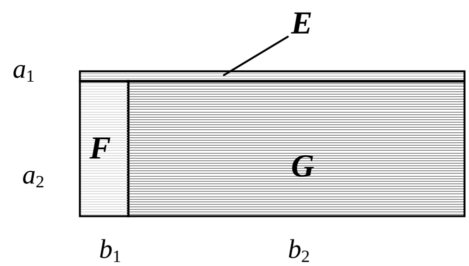
Let the subject's preferences be represented by (1) with $p_1 = p_2 = 1/2$, $q_{11} = q_{21} = 1/3$, $q_{12} = q_{22} = 2/3$, and $\alpha > 0$. Red and Yellow ∪ Black are measurable with respect to B, so the subject will bet on them as if she were risk neutral and assigned them probabilities 1/3 and 2/3, respectively. Because Yellow and Black are not measurable with respect to B, they will be regarded as more uncertain than Red, and the subject will be risk averse with respect to bets on them. Similarly Red ∪ Black and Red ∪ Yellow will regarded as more uncertain, and hence more risky to bet on, than Yellow ∪ Black.

With different parameters, the same utility function (1) can also be used to explain the Allais paradox. Let E, F, and G be mutually exclusive events whose probabilities are estimated to be 0.01, 0.10, and 0.89, respectively, and consider the familiar pair of choices among monetary acts:

	E	F	G
f	\$1M	\$1M	\$1M
g	\$0	\$5M	\$1M

f'	\$1M	\$1M	\$0
g'	\$0	\$5M	\$0

Let the events be decomposed into a 2×2 partition of states with the mapping $a_1 \rightarrow E$, $a_2 \cap b_1 \rightarrow F$, and $a_2 \cap b_2 \rightarrow G$, as shown in the following schematic diagram:



Thus, E (the "sucker" event in acts g and g') is modeled as being more uncertain than F or G.[1] In the preference function (1), let $p_1 = 1 - p_2 = 1/100$ and $q_1 = 1 - q_2 = 10/99$. With these "probability" assignments, which agree with the previously given estimates, the subject will prefer f over g if $\alpha^{-1} <$ \$226,040 but will nevertheless prefer g' over f' as long as $\alpha^{-1} >$ \$21,982. Hence, for a plausible range of risk tolerances with respect to A-measurable acts, the subject will display the typical preference reversal of the Allais paradox.

4 A model of partially separable preferences

The examples in the preceding section suggest a novel hypothesis about the character of non-expected-utility preferences, namely that the decision maker may satisfy the independence axiom selectively within

[1] Actually, the key to this example is that the event G is neither A-measurable nor B-measurable, hence it cannot be modeled by a decision tree in which the first chance node resolves G or ~G.

partitions of the state space whose elements have similar degrees of uncertainty. As such, she may behave like an expected-utility maximizer with respect to assets in the same "uncertainty class," while exhibiting higher degrees of risk aversion toward assets that are "more uncertain." For example, she might be risk neutral—or even risk seeking—toward casino gambles, moderately risk averse toward investments in the stock market, and highly risk averse when insuring against health or property risks.

To formalize this idea, let the state space consist of mn states with two logically independent partitions $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$. For concreteness, suppose that the partition A represents the set of "natural" states of the world while B represents the set of outputs of an objective randomization device used to construct a horse lottery or to elicit a utility function for money. Thus, A-measurable events are potentially ambiguous while B-measurable events are unambiguous. Let $\boldsymbol{w}$, $\boldsymbol{w}^*$, $\boldsymbol{z}$, $\boldsymbol{z}^*$, denote wealth distributions over states, i.e., monetary acts. For any event E and acts $\boldsymbol{w}$ and $\boldsymbol{z}$, let $E\boldsymbol{w} + (1\text{–}E)\boldsymbol{z}$ denote the act that agrees with $\boldsymbol{w}$ on E and agrees with $\boldsymbol{z}$ on E^c. Suppose that preferences among acts satisfy the following partition-specific independence axioms:

A-independence: $E\boldsymbol{w} + (1\text{–}E)\boldsymbol{z} \geq E\boldsymbol{w}^* + (1\text{–}E)\boldsymbol{z} \Leftrightarrow E\boldsymbol{w} + (1\text{–}E)\boldsymbol{z}^* \geq E\boldsymbol{w}^* + (1\text{–}E)\boldsymbol{z}^*$ for all acts $\boldsymbol{w}$, $\boldsymbol{w}^*$, $\boldsymbol{z}$, $\boldsymbol{z}^*$ and every A-measurable event E, and conditional preference $\boldsymbol{w} \geq_E \boldsymbol{w}^*$ is accordingly defined for such events.

B-independence: $F\boldsymbol{w} + (1\text{–}F)\boldsymbol{z} \geq_i F\boldsymbol{w}^* + (1\text{–}F)\boldsymbol{z} \Leftrightarrow F\boldsymbol{w} + (1\text{–}F)\boldsymbol{z}^* \geq_i F\boldsymbol{w}^* + (1\text{–}F)\boldsymbol{z}^*$ for all B-measurable acts $\boldsymbol{w}$, $\boldsymbol{w}^*$, $\boldsymbol{z}$, $\boldsymbol{z}^*$ and every B-measurable event F, where $\geq_i$ denotes conditional preference given element a_i of A.

In other words, the decision maker satisfies the independence axiom unconditionally with respect to A-measurable events and conditionally with respect to B-measurable acts and events within each element of A. Such a person will be said to have *partially separable* preferences. A-independence and B-independence are similar to the time-0 and time-1 substitution axioms of Kreps and Porteus (1979), adapted to a framework of choice under uncertainty rather than risk and stripped of their temporal interpretation.

Preferences that are partially separable (as well as weakly ordered, monotonic, and smooth) are represented by a function U having the nested-additive form:

$$(2) \qquad U(\boldsymbol{w}) = \sum_{i=1}^{m} u_i\Big(\sum_{j=1}^{n} v_{ij}(w_{ij})\Big)$$

where w_{ij} denotes wealth in state a_ib_j, and $\{u_i\}$ and $\{v_{ij}\}$ are twice-differentiable state-dependent utility functions. The corresponding risk neutral probabilities satisfy

$$\pi_{ij} = \frac{u_i'\Big(\sum_{k=1}^{n} v_{ik}(w_{ik})\Big) v_{ij}'(w_{ij})}{\sum_{h=1}^{m} \sum_{l=1}^{n} u_h'\Big(\sum_{k=1}^{n} v_{hk}(w_{hk})\Big) v_{hl}'(w_{hl})} .$$

The risk aversion matrix $\boldsymbol{R}$ is the sum of a diagonal matrix and a block-diagonal matrix, with $r_{ij,kl} = 0$ if $i \neq k$ and

$$r_{ij,il} \; \frac{u_i''\Big(\sum_{h=1}^{n} v_{ih}(w_{ih})\Big)}{u_i'\Big(\sum_{h=1}^{n} v_{ih}(w_{ih})\Big)} v_{il}'(w_{il}) + \big(v_{ij}''(w_{ij}) / v_{ij}'(w_{ij})\big) 1_{jl}$$

Let $\boldsymbol{s}$ and $\boldsymbol{t}$ denote m- and mn-vectors defined by:

$$s_i \; = \; \frac{u_i''\Big(\sum_{h=1}^{n} v_{ih}(w_{ih})\Big)}{u_i'\Big(\sum_{h=1}^{n} v_{ih}(w_{ih})\Big)} \sum_{h=1}^{n} v_{ih}'(w_{ih})$$

$$t_{ij} \; = \; v_j''(w_{ij})/v_j'(w_{ij}).$$

Let $\pi_i = \sum_{h=1}^{n} \pi_{ih}$ and $\pi_{h|i} = \pi_{ih}/\pi_i$ denote the induced marginal and conditional probabilities, and let $\bar{\boldsymbol{z}}$ be defined as the m-vector whose i^{th} element is $\bar{z}_i = \sum_{h=1}^{m} \pi_{h|i} z_{ih}$, i.e., the conditional risk-neutral expectation of $\boldsymbol{z}$ given event a_i. In these terms, we have

PROPOSITION:

For a decision maker with partially separable preferences, the risk premium for a neutral asset $\boldsymbol{z}$ is

$$\rho_b(\boldsymbol{z}) \quad \approx \quad \tfrac{1}{2} \sum_{i=1}^{m} \pi_i s_i \bar{z}_i^{\,2} \; + \; \tfrac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} \pi_{ij} t_{ij} z_{ij}^{\,2}$$

$$= \; \tfrac{1}{2} E_\pi[\boldsymbol{s}\,\bar{\boldsymbol{z}}^{\,2}] \; + \; \tfrac{1}{2} E_\pi[\boldsymbol{t}\,\boldsymbol{z}^{\,2}].$$

By comparison with the risk premium formula for separable preferences, it is suggestive to think of the second term on the RHS as a pure risk premium and the first term as an additional premium for the uncertainty surrounding A-measurable events, with $\boldsymbol{s}$ and $\boldsymbol{t}$ serving as vector-valued measures of aversion to uncertainty and risk, respectively. If $\boldsymbol{z}$ is neutral and A-measurable, then $z_{ij} \equiv \bar{z}_i$ and the total risk premium is $\tfrac{1}{2} E_\pi[(\boldsymbol{s} + \boldsymbol{t})\boldsymbol{z}^{\,2}]$. (Here $\boldsymbol{s} + \boldsymbol{t}$ is understood to be the vector whose ij^{th} element is $s_i + t_{ij}$.) If $\boldsymbol{z}$ is neutral and B-measurable while A and B are independent under $\boldsymbol{\pi}$, then $\bar{z}_i = E_\pi[\boldsymbol{z}] = 0$ for every i and the total risk premium is $\tfrac{1}{2} E_\pi[\boldsymbol{t}\,\boldsymbol{z}^2]$.

As a special case of (2), suppose that the component utility functions are state-independent expected utilities of the form $u_i(v) = p_i u(v)$ and $v_{ij}(x) = q_{ij} v(x)$, where $\boldsymbol{p}$ is a marginal probability distribution on A and $\boldsymbol{q}_i$ is a conditional probability distribution on B given a_i, yielding:

$$U(\boldsymbol{w}) = \sum_{i=1}^{m} p_i u\left(\sum_{j=1}^{n} q_{ij} v(w_{ij})\right). \tag{3}$$

Then the decision maker behaves as though she assigns probability $p_i q_{ij}$ to state $a_i b_j$ and she bets on events measurable with respect to A as though her utility function were $u(v(x))$. If A and B are also independent, i.e., if q_{ij} is the same for all i, she meanwhile bets on events measurable with respect to B as though her utility function for money were $v(x)$. If u is concave, she is uniformly more risk averse with respect to A-measurable bets than to B-measurable bets, suggesting that she regards A as more uncertain than B and is averse to uncertainty. Thus, concavity of v encodes the decision maker's aversion to risk while concavity of u encodes her aversion to the additional uncertainty surrounding the A-measurable events.

The preference model (3) will henceforth be called *partially separable utility* (PSU). For a decision maker with PSU preferences, a utility function for money elicited via choices among objectively-randomized lotteries cannot be used to predict or prescribe choices among natural lotteries, contrary to usual decision-analytic practice. Nevertheless, such a decision maker is perfectly rational in the sense that her behavior does not create opportunities for arbitrage, and she can still solve a decision tree by dynamic programming provided that all-but-the-last chance node on every path is A-measurable. Whether she is able to use dynamic programming in practice will depend on whether she frames a dynamic decision problem in such a way that the A-measurable events are resolved first.

If the PSU decision maker is further assumed to have constant (i.e., nonstochastic) prior wealth x, then her risk neutral distribution is the product $\pi_{ij} = p_i\, q_{ij}$ and her local attitude toward risk and uncertainty can be summarized by a scalar risk aversion measure $t(x) = -v''(x)/v'(x)$ and a scalar uncertainty aversion measure $s(x) = -u''(v(x))/u'(v(x))v'(x)$. Under these conditions, $\bar{\boldsymbol{z}}$ is the vector whose i^{th} element is $\bar{z}_i = \sum_{j=1}^{n} q_{ij} z_{ij}$, the conditional expectation of $\boldsymbol{z}$ given a_i under the distribution q_i. The total risk premium for a neutral asset $\boldsymbol{z}$ is then

$$\rho_b(\boldsymbol{z}) \quad \approx \quad ½\, s(x)E_p[\bar{\boldsymbol{z}}^2] + ½\, t(x)E_\pi[\boldsymbol{z}^2].$$

The measures $s(x)$ and $t(x)$ are convenient hyperbolic functions if u and v are utilities from the HARA (generalized log/power/exponential) family, as shown in the following table:

Table 1: risk and uncertainty aversion measures for HARA partially separable utilities

	$u(x)$	$v(x)$
(i)	$-\exp(-\alpha x)$	$(\mathrm{sgn}(\beta)/\beta)(x+\gamma)^\beta$
(ii)	$-\mathrm{sgn}(\alpha)\exp(-\alpha x)$	$\log(x+\gamma)$
(iii)	$(\mathrm{sgn}(\alpha)/\alpha)x^\alpha$	$(1/\beta)(x+\gamma)^\beta$
(iv)	$\log(x)$	$(1/\beta)(x+\gamma)^\beta$

	$s(x)$	$t(x)$	restrictions
(i)	$\alpha(x+\gamma)^{\beta-1}$	$(1-\beta)/(x+\gamma)$	$\alpha > 0,\ \beta \leq 1$
(ii)	$\alpha/(x+\gamma)$	$1/(x+\gamma)$	$\alpha > -1$
(iii)	$(1-\alpha)\beta/(x+\gamma)$	$(1-\beta)/(x+\gamma)$	$0<\beta\leq 1$, $\alpha<1/\beta$
(iv)	$\beta/(x+\gamma)$	$(1-\beta)/(x+\gamma)$	$\beta > 0$

Here, (ii) and (iv) are limiting cases of (i) and (iii) in which $\beta\to 0$ and $\alpha\to 0$, respectively, and wealth is assumed to be bounded below by $-\gamma$. The conditions in the last column imply $t(x) \geq 0$ and $s(x) + t(x) \geq 0$, ensuring convexity of preferences. Note that if $\alpha<0$ in (ii) or $\alpha>1$ in (iii), $s(x)$ is negative and the decision maker is *less* risk averse toward A-measurable events than B-measurable events. This could represent a situation in which the decision maker is uncertainty-loving (despite having convex preferences overall) *or* regards A as the less uncertain partition.

The preference model (1) introduced earlier is the special case of HARA partially separable utility in which $m = n = 2$, $u(x) = -\exp(-\alpha x)$, $v(x) = x$, $q_{11} = q_{21}$, and $q_{12} = q_{22}$. The same general construction can, of course, be extended to 3-way partitions, 4-way partitions, etc., all having different degrees of uncertainty, although the 2-way partition suffices to model the basic dichotomy between risk and uncertainty.

5 Second-order probabilities and utilities

In the discussion of partially separable preferences in the preceding section, the partitions A and B were interpreted to represent sets of observable, payoff-relevant events that were, respectively, ambiguous or unambiguous. In this section, a different interpretation of the same model will be suggested, namely that the partition B represents the observable, payoff-relevant events while the partition A represents possible *credal states* for the decision maker in which she may have different probabilities and/or utilities. The set of credal states could have various interpretations in practice. For example, it could be interpreted to represent uncertainty about the decision maker's state of mind after further introspection and/or learning has taken place, or it could be interpreted to represent *model* *risk*—i..e., uncertainty about the model which ought to be used for purposes of decision analysis.

Henceforth, let the wealth vector $\boldsymbol{w}$ be singly-subscripted, with w_j representing wealth in state $b_j \in B$, and consider the following specialization of (2):

$$(4) \qquad U(\boldsymbol{w}) = \sum_{i=1}^{m} p_i u_i (v_i^{-1}(\sum_{j=1}^{n} q_{ij} v_i (w_j))) ,$$

where $\boldsymbol{p}$ is a probability distribution on A and, for each i, $\boldsymbol{q_i}$ is a probability distribution on B. (The former $u_i(.)$ has been rewritten as $p_i\, u_i(v_i^{-1}(.))$, and the former $v_{ij}(.)$ has been assumed to have the form $q_{ij}v_i(.)$, which is conditionally state-independent.) The argument of u_i is now the *certainty equivalent* of $\boldsymbol{w}$ obtained from an expected-utility calculation with probability distribution $\boldsymbol{q_i}$ and utility function v_i:

$$CE_i(\boldsymbol{w}) = v_i^{-1}(\sum_{j=1}^{n} q_{ij} v_i (w_j)) ,$$

in terms of which the utility function (4) becomes:

$$(5) \qquad U(\boldsymbol{w}) = \sum_{i=1}^{m} p_i u_i (CE_i (\boldsymbol{w})) .$$

A utility function of essentially this same form was used by Segal (1989) to model behavior violating the reduction of compound lotteries axiom in two-stage lotteries under risk. Here the "first stage" lottery is the selection of an element a_i from partition A, which can be interpreted as a credal state within which the decision maker behaves like an expected-utility maximizer with probability distribution $\boldsymbol{q_i}$ and utility function v_i. The implication of (5) is that, prior to the resolution of the first-stage lottery, the decision maker is uncertain about her credal state (as represented by a second-order probability distribution $\boldsymbol{p}$) and is potentially averse to that uncertainty (as represented by a second-order utility function u which is applied to the certainty equivalents realized in different credal states). If $u_i = v_i$ for every i, then the second-order uncertainty about probabilities and utilities can be integrated out and the decision maker has (possibly state-dependent) expected-utility preferences and is uncertainty-neutral, but otherwise she has non-expected utility preferences and may be uncertainty averse.

The specific utility function (1), which was previously used to explain the Ellsberg and Allais paradoxes with partition A interpreted as a set of ambiguous (but observable) events, can now be re-interpreted as a special case of (4)-(5) in which the decision maker is an expected-value maximizer (i.e., risk neutral) within each credal state, but she is uncertain about her probability distribution and is averse to that uncertainty with a constant degree of uncertainty aversion quantified by α. In particular, the model of the 2-color Ellsberg paradox is a special case of (4) in which the decision maker thinks it is equally likely that the unknown urn contains all red balls or all black balls, while she is certain that the known urn contains equal numbers of red and black balls. There are four payoff-relevant events: b_1 = RR, b_2 = RB, b_3 = BR, b_4 = BB; the decision maker's two possible credal states are represented by first-order probability distributions $\boldsymbol{q_1}$ = (½, ½, 0, 0) and $\boldsymbol{q_2}$ = (0, 0, ½ , ½); her first-order utility is linear, $v_i(x) \equiv x$; and her second-order probabilities and utilities are $\boldsymbol{p}$ = (½, ½) and $u_i(x) \equiv -\exp(-\alpha x)$. With these parameter assignments, the decision maker is risk neutral with respect to bets on the known urn and risk averse with respect to bets on the unknown urn, exactly as before. A two-stage lottery interpretation of the Ellsberg paradox was also given by Segal (1987), although there the underlying utility model was that of anticipated (rank-dependent) utility rather than expected utility.

Similarly, the model of the 3-color Ellsberg paradox is a special case of (4) in which the decision maker thinks it is equally likely that the urn contains 60 yellow balls and zero black balls or vice versa, while she is certain that it also contains 30 red balls. There are now three observable events, namely b_1 = Red, b_2 = Yellow, and b_3 = Black, and two credal states represented by the probability distributions $\boldsymbol{q_1}$ = (1/3, 2/3, 0) and $\boldsymbol{q_2}$ = (1/3, 0, 2/3) over those events; and the credal states are considered equally likely, i.e., $\boldsymbol{p}$ = (½, ½). When these values are substituted into (4), together with $v_i(x) \equiv x$ and $u_i(x) \equiv -\exp(-\alpha x)$, the result is the same as (1) with the values previously used for the 3-color paradox. As noted earlier, the model for the 3-color Ellsberg paradox can also be adapted to explain the Allais paradox, which exhibits a similar direct violation of the independence axiom. In the Allais example, the three observable events are $b_1 = E$, $b_2 = F$, and $b_3 = G$, and the preference model given earlier is equivalent to (4) with two credal states represented by probability distributions $\boldsymbol{q_1}$ = (1, 0, 0) and $\boldsymbol{q_2}$ = (0, 10/99, and 89/99), having second-order probabilities $\boldsymbol{p}$ = (1/100, 99/100). The latter model implies the following interpretation of the Allais paradox: the subject thinks that the game is rigged so that one alternative is dominant over the other in both pairs, she just doesn't know which one. In particular, she thinks there is a 1% chance that E is sure to happen, in which case f and f' are strictly dominant, and conversely there is a 99% chance that E is sure not to happen, in which case g and g' are weakly dominant.

Levi (1986 and elsewhere) has suggested a fundamentally different interpretation of the Allais paradox, namely that it is due to indeterminacy of utilities rather than probabilities. That interpretation can also be accomodated by the present model, although here a second-order probability distribution is assessed over the set of credal states (possible utility functions), whereas in Levi's model alternatives are compared on the basis of admissibility criteria referring to extremal utilities. In the Allais example, suppose that the decision maker is certain that events E, F, and G have the given probabilities of 0.01, 0.10, and 0.89, respectively, but meanwhile she is uncertain about her utility function. In particular, suppose that she has an exponential utility function whose risk aversion parameter is equally likely to be 1 or 10, when payoffs are measured in \$M. In other words, her risk tolerance (the reciprocal of her risk aversion coefficient) is equally likely to be \$100,000 or \$1,000,000. Furthermore, assume $u_i(x) \equiv x$ in (4), so that the decision maker evaluates alternatives on the basis of the second-order expectations of their first-order certainty equivalents. For such a decision maker, the certainty equivalent of f is \$1M, while the certainty equivalent of g is equally likely to be \$0.46M or \$1.08M, whose expected value is \$0.77M. Hence f is preferred to g. Meanwhile, the certainty equivalent of f' is equally likely to be \$0.012M or \$0.072M, yielding an expected value of \$0.042M, while the certainty equivalent of g' is equally likely to be \$0.011M or \$0.105M, yielding an expected value of \$0.058M, hence g' is preferred to f'.

Seidenfeld (1986) has shown that violations of independence in sequential decisions under risk can lead to *sequential incoherence*. The preference model presented in this paper refers to static decisions under uncertainty and cannot, per se, lead to sequential incoherence. When faced with a sequential decision problem, a decision maker with partially separable preferences could either solve the problem in normal form and proceed as though "risks borne but not realized" were relevant (Machina 1989), or, more interestingly, she might regard her future decisions as stochastic due to her uncertain credal state.

6 Comparison with other preference models

This section compares the partially-separable preference model against other well-known preference models. First, as already noted, two-stage utility functions have previously been used by Kreps and Porteus (1979) and Segal (1989) to model preferences for temporal or compound lotteries under conditions of risk (known probabilities). Here, the setting is that of uncertainty—i.e., states of nature with subjectively determined probabilities—and it is not necessary to think of the decision problem as having a temporal or compound structure, though the uncertain-credal-state interpretation could perhaps be viewed in temporal terms.

Second-order probabilities have often been used in Bayesian statistical models to represent imprecise prior distributions, although in those models the second-order uncertainty has no behavioral implications: it can always be integrated out to yield an equivalent representation of preferences in terms of first-order expected utility. In constrast, the partially-separable-preference model admits the possibility of a second-order utility function which captures aversion to uncertainty and/or it admits uncertainty in the first-order utility, thereby rationalizing behavior that is inconsistent with standard Bayesian theory. In "quasi-Bayesian" models, incomplete preferences are represented by sets of probabilities and/or utilities. Here, the preference ordering is "completed" through the use of second-order probabilities and utilities.

Epstein (1999) has defined uncertainty aversion in relative terms by reference to sets of ambiguous and unambiguous acts, with *probabilistic sophistication* (Machina and Schmeider 1992) serving as a benchmark for uncertainty neutrality. (A decision maker is probabilistically sophisticated if there is a probability distribution on states such that her preferences among acts depend only on the probability distributions they induce on consequences, regardless of whether she is an expected-utility maximizer.) Epstein's definition of uncertainty aversion, like that of probabilistic sophistication, applies to a Savage-act framework in which the primitive rewards are abstract consequences whose utilities are effectively state-independent, providing a basis for extracting personal probabilities from preferences among acts. The analysis in this paper, in constrast, applies to a state-preference framework in which the primitive rewards are quantities of money whose utilities may be state-dependent and hence inseparable from subjective probabilities, rendering it impossible to apply the Machina-Schmeidler definition of probabilistic sophistication in a completely general manner. Nevertheless, a sufficient condition for probabilistic sophistication in the state-preference framework is that the decision maker's preferences should be fully additively separable, because such preferences have a state-dependent expected-utility representation, even if the probabilities are not unique. Using the latter benchmark, a decision maker whose preferences are represented by (4) is uncertainty averse by Epstein's definition if $u_i(v_i^{-1}(.))$ is concave for every i. To show this, suppose that there is a decision maker whose preferences are represented by (3) with $\{\boldsymbol{q}_i\}$ distinct and $u_i(v_i^{-1}(.))$ concave for every i. For such a person, an act $\boldsymbol{w}$ is unambiguous if the first-order expected utility

$$\sum_{j=1}^{n} q_{ij} v_i \, (w_{ij})$$

is the same in every credal state i, and it is ambiguous otherwise. (Note that the definition of an ambiguous act is subjective and particular to the decision maker. For example, the contents of Ellsberg's urn might be known to the experimenter but not to the subject.) Now consider a second decision maker whose preferences have the same representation except that, for the second decision maker, $u_i = v_i$ and hence $u_i(v_i^{-1}(.))$ is linear for every i. Then the second decision maker is uncertainty neutral—i.e., she has fully additively separable utility and is therefore probabilistically sophisticated. The two decision makers then assign the same first-order expected utility to every act, and the second decision maker evaluates the first-order expected utilities in a risk-neutral manner (by taking expectations with respect to the second-order distribution), while the first decision maker evaluates them in a risk-averse manner (using the same second-order probabilities together with a concave second-order utility function). Hence, the two decision makers will assign the same overall certainty equivalents to every unambiguous act but the first decision maker will assign lower certainty equivalents to ambiguous acts.

The characterization of risk and uncertainty aversion in this paper applies to general locally-smooth preferences, which differ from the Choquet expected utility preferences that are currently the most popular alternative to subjective expected utility (Schmeidler 1989; see also Epstein 1999). The differences between the two types of preferences are transparent and have testable implications. Choquet expected utility preferences are the same as subjective expected utility preferences within each comonotonic set, which is a convex cone in state-payoff space. For example, in two dimensions, the comonotonic sets are the half-planes above and below the line $x = y$. In three dimensions, the comonotonic sets are six wedges whose cutting edges meet along the line $x = y = z$. Within each such cone, the decision maker's indifference curves and risk neutral probabilities are determined by a fixed subjective probability distribution and a state-independent utility function. At the boundaries between cones, the indifference curves are kinked: the subjective probabilities jump to new (usually more pessimistic) values while the marginal utilities remain the same, so the risk neutral probabilities also change discontinuously.

Several details are important. First, the Choquet model requires knowledge of prior wealth in order to determine the comonotonic sets, and constant acts play an even more critical role than they do in the standard theory. All of the usual caveats about the difficulties of observing prior wealth and defining constant acts under naturalistic conditions therefore apply. Second, a Choquet expected utility maximizer displays true uncertainty aversion *only* when comparing prospects that lie in different comonotonic sets. She is "locally risk averse but uncertainty neutral" and uses a state-independent local Pratt-Arrow measure to compute risk premia for small gambles, ambiguous or otherwise, *except* when her prior wealth happens to lie on the boundary between two comonotonic sets (e.g., in an idealized state of constant prior wealth). Whereas, under a general smooth preference model such as the partially separable model introduced here, a decision maker may be locally uncertainty averse *everywhere* in state-payoff space. Third, when a Choquet expected utility maximizer finds herself on the boundary between two comonotonic sets, she exhibits *first-order* risk aversion. In other words, she is risk averse even for infinitesimal gambles. The Choquet model offers one way to model behavior that is first-order risk averse, but not the only way. For example, a subject who has incomplete (partially ordered) preferences will exhibit first-order risk aversion everywhere.

The empirical questions, then, are: (i) whether uncertainty aversion is a first-order or second-order phenomenon, and (ii) whether it affects *all* choices that involve uncertain events or only choices between alternatives that induce different rankings of states. More specifically, is uncertainty aversion revealed by valuations of small assets *only* when prior wealth is constant across states? The CEU model localizes uncertainty-averse behavior on the boundaries of comonotonic sets, which just happens to be where the empirical light shines the brightest. It is easiest to demonstrate violations of Savage's axioms in choices among simple acts that lead to only two or three distinct consequences with state-independent valuations—e.g., a status quo and one or two prizes—which do not turn on subtle issues of cardinal utility measurement. Practically the only non-trivial choices under such conditions are those in which the acts lie in different comonotonic sets. By comparison, it is rather hard to elicit violations of SEU in choices among acts in the relative interior of the same comonotonic set, because such choices depend sensitively on many cardinal utilities. Nevertheless, it is intuitively plausible that in the Ellsberg urn problem, a subject might "feel" differently toward the two urns regardless of the complexity of the acts pegged to them.

The following hypothetical experiment illustrates the possibility—as well as the difficulty—of eliciting violations of the independence axiom in choices among comonotonic acts. Consider again a two-urn problem in which urn 1 contains equal numbers of red and black balls and urn 2 contains red and black in unknown proportions. Suppose the subject's preferences are assessed for the following two pairs of bets: (i) win \$100 if the ball drawn from urn 1 is Red ("R1") vs. win \$100 if the ball drawn from urn 2 is Red, ("R2"), and (ii) win \$100 if the ball drawn from urn 1 is Black ("B1") vs. win \$100 if the ball drawn from urn 2 is Black, ("B2"). Furthermore, suppose that the subject is endowed with the following distribution of prior wealth:

		Urn 2	
		Red (??)	Black (??)
Urn 1	Red (1/2)	\$0	\$200
	Black (1/2)	\$300	\$100

Thus, the decision maker's prior expected wealth is \$150 regardless of the proportions of balls in urn 2. Against this background, the four bets are comonotonic. If the subject nevertheless prefers to bet on the ball drawn from the known urn regardless of the winning color—i.e., if R1>R2 and B1>B2—then she violates CEU but still could conform to the PSU model.[2]

7 Discussion

The resurrection of cardinal utility theory by von Neumann–Morgenstern and Savage was predicated on the argument that, under conditions of risk and uncertainty, preferences should be separable across

[2] In a pilot experiment with students at Duke University, involving \$10's rather than \$100's, a slight majority of subjects exhibited this pattern.

mutually exclusive events. Although separability of risk preferences does seem reasonable in many situations, at least as a simplifying assumption, it is no longer accepted as a universal normative or descriptive principle. The currently-most-popular alternative theory, Choquet expected utility, admits a special kind of inseparability by positing that indifference curves are kinked at the boundaries of comonotonic sets—the so-called "45-degree certainty line" in state-payoff space—while conforming to subjective expected utility theory everywhere else. In giving a central role to the 45-degree uncertainty line, the CEU model depends heavily on some other assumptions of subjective expected utility theory that are equally questionable (Shafer 1986), namely the assumptions that subjective probabilities (additive or otherwise) are uniquely determined by preferences and that it is always possible to identify a set of riskless acts which have constant consequences for the decision maker.

This paper has presented a simpler alternative model of non-expected-utility preferences that does not require kinked indifference curves nor the unique determination of subjective probabilities nor the identification of riskless acts. The decision maker is permitted to display different degrees of risk aversion toward different partitions of states of nature, which leads to a simple characterization of aversion to uncertainty, viz., the decision maker is uncertainty averse if she is more risk averse toward ambiguous events than unambiguous ones. Equivalently, she may behave as though her credal state is uncertain and she is averse to the credal uncertainty. A decision maker may be uncertainty averse by this definition and yet have additive hierarchical probabilities for all events and conform to subjective expected utility theory within a subalgebra of events having the same degree of ambiguity or within a given credal state. This preference model does not necessarily invalidate conventional methods of decision analysis—rather, it suggests a simple way that decision analysis could be extended to account for model risk—but it does cast doubt on the common practice of assessing utility functions for naturalistic decisions by contemplating bets on objective randomization devices.

Acknowledgements

This research was supported by the National Science Foundation under grant 98-09225 and by the Fuqua School of Business. I am grateful for the comments of two anonymous referees.

References

Arrow, K.J. (1964) "The Role of Securities in the Optimal Allocation of Risk-Bearing." *Quarterly J. Economics* **31**, 91—96

Debreu, G. (1959) *Theory of Value*. Cowles Foundation, New Haven

Debreu, G. (1960) "Topological Methods in Cardinal Utility Theory" in K.J. Arrow, S. Karlin, & P. Suppes (eds.), *Mathematical Methods in the Social Science*s, 16-26, Stanford University Press, Stanford, CA.

Ellsberg, D. (1961) "Risk, Ambiguity and the Savage Axioms," *Quarterly J. Economics* **75**, 643—669.

Epstein, L. (1999) "A Definition of Uncertainty Aversion." *Review of Economic Studies* **66**, 579—608

Hirshleifer, J. (1965) "Investment Decision Under Uncertainty: Choice-Theoretic Approaches." *Quarterly J. Economics* **74**, 509—536

Kreps, D. and E. Porteus (1979) "Temporal Von Neumann-Morgenstern and Induced Preferences." *J. Econ. Theory* **20**, 89—109.

Levi, I. (1986) "The Paradoxes of Ellsberg and Allais." *Economics and Philosophy* **2**, 23—53

Machina, M. (1989) "Dynamic Consistency and Non-Expected Utility Models of Choice Under Uncertainty." *J. Econ. Lit.* **27**, 1622-1688

Machina, M.J. & D. Schmeidler (1992) "A More Robust Definition of Subjective Probability." *Econometrica* **60**, 745—780

Nau, R.F. (2001) "A Generalization of Pratt-Arrow Measure to Non-expected-utility Preferences and Inseparable Probabilities and Utilities," working paper, Fuqua School of Business

Pratt, J.W. (1964) "Risk Aversion in the Small and in the Large," *Econometrica* **32**, 122—136.

Schervish, M.J., T. Seidenfeld, and J.B. Kadane (1990) "State-Dependent Utilities." *J. American Statistical Association* **85**, 840—847.

Schmeidler, D. (1989) Subjective Probability and Expected Utility Without Additivity." *Econometrica* **57**, 571—588

Seidenfeld, T. (1986) "Decision Theory Without 'Independence' or Without 'Ordering': What is the Difference?" *Economics and Philosophy* **2**, 267—290

Segal, U. (1987) "The Ellsberg Paradox and Risk Aversion: An Anticipated Utility Approach." *Intl. Econ. Rev.* **28**, 175—202

Segal, U. (1989) "Two-Stage Lotteries Without the Reduction Axiom." *Econometrica* **58**, 349—377

Segal, U. and A. Spivak (1990) "First Order Versus Second Order Risk Aversion." *J. Econ. Theory* **51**, 111—125

Shafer, G. (1986) "Savage Revisited." *Statistical Science* **1**, 463-501

Walley, P. (1991). *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London.

Yaari, M. (1969) "Some Remarks on Measures of Risk Aversion and their Uses," *J. Econ. Theory* **1**, 315—329